



On the properties of forward and backward shifts of vector fields

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Abstract. The paper investigates some properties of recently defined forward and backward shifts of vector fields. The main purpose of the paper is to show that the forward and backward shifts of vector fields commute with the Lie bracket operator and with some commonly used system transformations. The latter include, for example, classical and parametrized state transformations as well as static and dynamic state feedbacks. These properties become important when studying control problems involving such transformations.

Keywords: control theory, nonlinear control system, discrete-time system, algebraic approach, forward and backward shifts, vector fields, commutativity.

1. INTRODUCTION

In control theory different mathematical approaches are being used to analyze and design control systems. Perhaps the most popular approach is based on differential geometry, see for instance [7,15]. The advantage of this approach is that the objects and tools can be easily interpreted geometrically. Another widely-used approach is based on the vector spaces of differential 1-forms defined over a suitable field of meromorphic functions, see for instance [1,2,5]. This method is more suitable to study generic properties as well as in cases when finite extensions are needed, for example, when dynamic feedback is searched. Moreover, the approach based on differential 1-forms is intuitively easier to understand, since it has more similarities with the linear case. It is important to have different approaches available so that one has a choice when studying a specific problem with definite needs and assumptions.

In [14] the algebraic approach of 1-forms [1] was extended to also address vector fields. Instead of vector space of 1-forms, an infinite dimensional vector space (over the field of meromorphic functions \mathcal{H}) of vector fields, denoted by \mathcal{E}^* , was constructed in analogy with the vector space of 1-forms. Moreover, the operators of forward and backward shifts of vector fields were introduced with explicit formulas for their computation. The new concepts were used to re-address the iconic control problems such as accessibility and static state feedback linearization for systems being not drift invertible. However, the paper [14] did not explore the developed concepts further. For example, it is unclear how the vector field transforms under

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different system transformations, such as standard state transformation, regular static state feedback, dynamic state feedback and parametrized state transformation, used for instance in [8,9]. It is also important to study the interaction of these transformations with the new concepts of forward and backward shifts of vector fields. Another concept that needs to be considered when addressing numerous control problems is the Lie bracket of vector fields. Of course, standard definitions/rules exist in the n -dimensional space over \mathbb{R} , where n is the state dimension.

Note that in the continuous-time case many researchers have worked with infinite extensions (prolongations) of vector fields, e.g., [6,11] and the references therein. Since in jet spaces any function depends locally only on finitely many coordinates, the ordinary formula for Lie bracket can be applied without a problem (because the product terms in the Lie bracket formula will contain only finite sums and not an infinite series). However, there are some issues with applying Lie brackets, especially regarding integrability. For one thing, the Frobenius theorem does not generally hold in jet spaces [11].

Therefore, in this paper we further develop the approach introduced in [14]. The results of this paper are of technical nature, building the foundation for future studies. First, some new properties of the forward and backward shifts of vector fields are proved. Then the standard definition of the Lie bracket and its properties are incorporated to the approach. Most importantly, we show that the Lie bracket commutes with both forward and backward shifts. Unlike the standard differential geometric case, where the vector fields are defined over \mathbb{R} and the Lie bracket is an \mathbb{R} -bilinear map, here vector fields are defined over \mathcal{K} , but the defined Lie bracket is not a \mathcal{K} -bilinear map. Therefore, some results on the involutivity of vector spaces of vector fields are revisited. Finally, we study the properties of different system transformations and show that the most commonly used transformations commute with forward and backward shifts as well as with the Lie bracket.

The algebraic approach introduced in [14] and developed further in the current paper has been already used to study structural problems like accessibility and feedback linearization [14], linearization only by state transformation [12] and transforming a discrete-time system into a classical observer form [13]. These results generalize the existing ones from reversible to more general case when only submersivity is assumed. The concept of the Lie bracket and different system transformations were used in [12,13] without formally introducing them within the algebraic approach from [14]. Moreover, some results proved in this paper were shown to hold in [12], but only in some very special cases. Commutativity of different operators and transformations, proved in the current paper, simplifies the future proofs if the tools from [14] will be used to solve different problems, since the order of applying different operators is not important anymore. One such problem under study is transformation of the state equations into the extended observer form using the parametrized state transformations. The extended observer form allows greatly to enlarge the class of systems admitting observers with linear error dynamics.

As for other current studies on discrete-time nonlinear control systems, using the approach based on vector fields, we refer to [10,16,17]. These papers rely on differential geometric techniques whereas our approach is purely algebraic. Both techniques speak different languages and have roots in different theoretical approaches, which makes their precise comparison difficult. Whereas we address time invariant systems, the papers [16,17] study more general time-varying systems. Moreover, in geometric approach the span of vector fields is taken over the ring of smooth functions, whereas in the algebraic framework, in contrast, we take the span over the field of meromorphic functions. The latter means that the results in [10,16,17] are local (around the fixed system trajectory), but in the algebraic approach the results are generic and hold almost everywhere. As for earlier paper [4] based on differential geometric approach, some results are given in [13] where comparison is done within the study of specific problem of transformation of state equations into the observer form.

The paper is organized as follows. Section 2 recalls the basics of the developed approach, including the forward and backward shifts of vector fields. Section 3 is devoted to the Lie bracket and its properties and Section 4 explores different system transformations and their relationship with forward and backward shifts of vector fields and with the Lie bracket. The paper ends with concluding remarks.

2. PRELIMINARIES

Let us recall the main concepts of the algebraic approach introduced in [14].

Consider the discrete-time nonlinear system

$$x^{(1)}(t) = \Phi(x(t), u(t)), \tag{1}$$

where $x^{(1)} = x(t + 1)$, $t \in \mathbb{Z}$, the vector $x(t) = (x_1(t), \dots, x_n(t))^T$ belongs to $X \subset \mathbb{R}^n$, the vector $u(t) = (u_1(t), \dots, u_m(t))^T$ belongs to $U \subset \mathbb{R}^m$ and the state transition map $\Phi : X \times U \rightarrow X$ is supposed to be meromorphic. Both X and U are assumed to be open in respective variables. Also, we assume that the map $\Phi = (\Phi_1, \dots, \Phi_n)^T$ can be extended to $\bar{\Phi} = (\Phi^T, \chi^T)^T : X \times U \rightarrow X \times \mathbb{R}^m$ by $\chi = (\chi_1, \dots, \chi_m)^T$ so that $\bar{\Phi}$ has a global analytic inverse, defined on its image. Let $z = \chi(x, u)$ and denote by $(\Lambda^T, \lambda^T)^T$, where $\Lambda = (\Lambda_1, \dots, \Lambda_n)^T$ and $\lambda = (\lambda_1, \dots, \lambda_m)^T$, the inverse of $\bar{\Phi}$, whereas $x = \Lambda(x^{(1)}, z)$, $u = \lambda(x^{(1)}, z)$.

Denote by \mathcal{K} the field of meromorphic functions of a finite number of variables from the set $\mathcal{C} = \{x_i, u_j^{(k)}, z_q^{(-l)}; i = 1, \dots, n; j, q = 1, \dots, m; k \geq 0; l > 0\}$. Define the forward-shift operator $\sigma : \mathcal{K} \rightarrow \mathcal{K}$ as follows. Let $x^\sigma = \Phi(x, u)$, $(u^{(k)})^\sigma = u^{(k+1)}$, $(z^{(-l)})^\sigma = z^{(-l+1)}$ for $l \geq 2$, $(z^{(-1)})^\sigma = \chi(x, u)$ and

$$(\varphi(x, u, \dots, u^{(k)}, z^{(-1)}, z^{(-2)}, \dots, z^{(-l)}))^\sigma = \varphi(\Phi(x, u), u^{(1)}, \dots, u^{(k+1)}, \chi(x, u), z^{(-1)}, \dots, z^{(-l)}).$$

We also define the backward-shift operator $\rho : \mathcal{K} \rightarrow \mathcal{K}$ as the inverse of σ . One has $x^\rho = \Lambda(x, z^{(-1)})$, $u^\rho = \lambda(x, z^{(-1)})$, $(u^{(k)})^\rho = u^{(k-1)}$ for $k \geq 1$, $(z^{(-l)})^\rho = z^{(-l-1)}$ and

$$(\varphi(x, u, u^{(1)}, \dots, u^{(k)}, z^{(-1)}, \dots, z^{(-l)}))^\rho = \varphi(\Lambda(x, z^{(-1)}), \lambda(x, z^{(-1)}), u, \dots, u^{(k-1)}, z^{(-2)}, \dots, z^{(-l-1)}).$$

Since σ is an automorphism of \mathcal{K} , then the pair (\mathcal{K}, σ) is an inversive difference field.

Let $\mathcal{E} := \text{span}_{\mathcal{K}}\{d\varphi | \varphi \in \mathcal{K}\}$ be the vector space spanned over \mathcal{K} by the differentials of the elements of \mathcal{K} . The elements of \mathcal{E} are called 1-forms and can be written in the form

$$\omega = \sum_{i=1}^n A_i dx_i + \sum_{k \geq 0} \sum_{j=1}^m B_{jk} du_j^{(k)} + \sum_{l > 0} \sum_{q=1}^m C_{ql} dz_q^{(-l)}. \tag{2}$$

The forward- and backward-shift operators σ and ρ can be extended to \mathcal{E} in a natural way by applying these operators to all the functions appearing in (2), i.e.,

$$\begin{aligned} \omega^\sigma &= \sum_{i=1}^n A_i^\sigma dx_i^\sigma + \sum_{k \geq 0} \sum_{j=1}^m B_{jk}^\sigma d(u_j^{(k)})^\sigma + \sum_{l > 0} \sum_{q=1}^m C_{ql}^\sigma d(z_q^{(-l)})^\sigma \\ &= \sum_{i=1}^n \left(\sum_{\kappa=1}^n A_\kappa^\sigma \frac{\partial \Phi_\kappa}{\partial x_i} + \sum_{\alpha=1}^m C_{\alpha 1}^\sigma \frac{\partial \chi_\alpha}{\partial x_i} \right) dx_i + \sum_{r=1}^m \left(\sum_{b=1}^n A_b^\sigma \frac{\partial \Phi_b}{\partial u_r} + \sum_{\beta=1}^m C_{\beta 1}^\sigma \frac{\partial \chi_\beta}{\partial u_r} \right) du_r \\ &+ \sum_{k \geq 0} \sum_{j=1}^m B_{jk}^\sigma du_j^{(k+1)} + \sum_{l > 1} \sum_{q=1}^m C_{ql}^\sigma dz_q^{(-l+1)} \end{aligned}$$

and

$$\begin{aligned} \omega^\rho &= \sum_{i=1}^n A_i^\rho dx_i^\rho + \sum_{k \geq 0} \sum_{j=1}^m B_{jk}^\rho d(u_j^{(k)})^\rho + \sum_{l > 0} \sum_{q=1}^m C_{ql}^\rho d(z_q^{(-l)})^\rho \\ &= \sum_{i=1}^n \left(\sum_{\kappa=1}^n A_\kappa^\rho \frac{\partial \Lambda_\kappa}{\partial x_i} + \sum_{\alpha=1}^m B_{\alpha 0}^\rho \frac{\partial \lambda_\alpha}{\partial x_i} \right) dx_i + \sum_{k \geq 1} \sum_{j=1}^m B_{jk}^\rho du_j^{(k-1)} \\ &+ \sum_{r=1}^m \left(\sum_{b=1}^n A_b^\rho \frac{\partial \Lambda_b}{\partial z_r^{(-1)}} + \sum_{\beta=1}^m B_{\beta 0}^\rho \frac{\partial \lambda_\beta}{\partial z_r^{(-1)}} \right) dz_r^{(-1)} + \sum_{l > 0} \sum_{q=1}^m C_{ql}^\rho dz_q^{(-l-1)}. \end{aligned}$$

Define the space \mathcal{E}^* dual to \mathcal{E} whose elements are vector fields in the form

$$f = \sum_{i=1}^n \tilde{f}_i \frac{\partial}{\partial x_i} + \sum_{k \geq 0} \sum_{j=1}^m \tilde{f}_{jk} \frac{\partial}{\partial u_j^{(k)}} + \sum_{l > 0} \sum_{q=1}^m \hat{f}_{ql} \frac{\partial}{\partial z_q^{(-l)}}, \tag{3}$$

where a finite number of coefficients \tilde{f}_i , \tilde{f}_{jk} and \hat{f}_{ql} are non-zero. For every vector field (3) let $K(L)$ be such that $\tilde{f}_{jK} \neq 0$ ($\hat{f}_{qL} \neq 0$) and $\tilde{f}_{jk} \equiv 0$ ($\hat{f}_{ql} \equiv 0$) for every $j = 1, \dots, m$ and $k > K$ ($q = 1, \dots, m$ and $l > L$). The duality between \mathcal{E}^* and \mathcal{E} is given by $\langle dx_i, f \rangle = \tilde{f}_i$, $\langle du_j^{(k)}, f \rangle = \tilde{f}_{jk}$ and $\langle dz_q^{(-l)}, f \rangle = \hat{f}_{ql}$. Additionally, $\langle \omega, \frac{\partial}{\partial x_i} \rangle = A_i$, $\langle \omega, \frac{\partial}{\partial u_j^{(k)}} \rangle = B_{jk}$ and $\langle \omega, \frac{\partial}{\partial z_q^{(-l)}} \rangle = C_{ql}$. Note that a vector field f can also be interpreted as a column vector $f = (\tilde{f}_1, \dots, \tilde{f}_n, \tilde{f}_{10}, \dots, \tilde{f}_{mK}, \hat{f}_{1,1}, \dots, \hat{f}_{mL})^T$ for some $K, L \in \mathbb{N}$, just as a 1-form ω can be interpreted as a row vector $\omega = (A_1, \dots, A_n, B_{10}, \dots, B_{mK}, C_{11}, \dots, C_{mL})$ for some $K, L \in \mathbb{N}$.

Definition 1. Given the vector field (3), its forward-shift is another vector field

$$f^\sigma = \sum_{i=1}^n \tilde{\alpha}_i \frac{\partial}{\partial x_i} + \sum_{k \geq 0} \sum_{j=1}^m \tilde{\beta}_{jk} \frac{\partial}{\partial u_j^{(k)}} + \sum_{l > 0} \sum_{q=1}^m \tilde{\gamma}_{ql} \frac{\partial}{\partial z_q^{(-l)}}, \tag{4}$$

which satisfies $\langle \omega, f \rangle^\sigma = \langle \omega^\sigma, f^\sigma \rangle$ for all $\omega \in \mathcal{E}$.

Definition 2. Given the vector field (3), its backward-shift is another vector field

$$f^\rho = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i} + \sum_{k \geq 0} \sum_{j=1}^m \beta_{jk} \frac{\partial}{\partial u_j^{(k)}} + \sum_{l > 0} \sum_{q=1}^m \gamma_{ql} \frac{\partial}{\partial z_q^{(-l)}}, \tag{5}$$

which satisfies $\langle \omega, f \rangle^\rho = \langle \omega^\rho, f^\rho \rangle$ for all $\omega \in \mathcal{E}$.

Proposition 1. [14]

- (i) The coefficients of the vector field f^σ are given by $\tilde{\alpha}_i = \langle dx_i^\rho, f \rangle^\sigma$, $\tilde{\beta}_{j0} = \langle du_j^\rho, f \rangle^\sigma$, $\tilde{\beta}_{jk} = \langle du_j^{(k-1)}, f \rangle^\sigma$ for $k > 0$, and $\tilde{\gamma}_{ql} = \langle dz_q^{(-l-1)}, f \rangle^\sigma$ for $l > 0$.
- (ii) The coefficients of the vector field f^ρ are given by $\alpha_i = \langle d\Phi_i, f \rangle^\rho$, $\beta_{jk} = \langle du_j^{(k+1)}, f \rangle^\rho$ for $k \geq 0$, $\gamma_{q1} = \langle d\chi_q, f \rangle^\rho$ and $\gamma_{ql} = \langle dz_q^{(-l+1)}, f \rangle^\rho$ for $l \geq 2$.

Proposition 2. For any $a \in \mathcal{X}$ and $f, g \in \mathcal{E}^*$ one has

- (i) $(f + g)^\sigma = f^\sigma + g^\sigma$ and $(f + g)^\rho = f^\rho + g^\rho$,
- (ii) $(af)^\sigma = a^\sigma f^\sigma$ and $(af)^\rho = a^\rho f^\rho$.

Proof: (i) From (i) of Proposition 1 one concludes that the coefficients of $(f + g)^\sigma$ are in the form

$$\langle d\xi^\rho, f + g \rangle^\sigma = \langle d\xi^\rho, f \rangle^\sigma + \langle d\xi^\rho, g \rangle^\sigma$$

for some $d\xi \in \{dx_i, du_j^{(k+1)}, dz_q^{(-l+1)}\}$. Since $\langle d\xi^\rho, f \rangle^\sigma$ is the corresponding coefficient of f^σ and, similarly, $\langle d\xi^\rho, g \rangle^\sigma$ is the corresponding coefficient of g^σ , then (i) must be true. The property $(f + g)^\rho = f^\rho + g^\rho$ is proved similarly.

(ii) From (i) of Proposition 1 one concludes that the coefficients of $(af)^\sigma$ are in the form

$$\langle d\xi^\rho, af \rangle^\sigma = a^\sigma \langle d\xi^\rho, f \rangle^\sigma$$

for some $d\xi \in \{dx_i, du_j^{(k+1)}, dz_q^{(-l+1)}\}$. Thus, (ii) must be true. The property $(af)^\rho = a^\rho f^\rho$ is proved similarly. \square

3. LIE BRACKET

In this section we introduce a Lie bracket in an algebraic manner and prove some of its properties. Note that the Lie bracket is defined here in a manner which does not make \mathcal{E}^* a (Lie) algebra over \mathcal{H} , even though in addition to its vector space structure it possesses a product that is a map from $\mathcal{E}^* \times \mathcal{E}^*$ to \mathcal{E}^* , taking the pair of vector fields (f, g) to the vector field $[f, g]$. Nevertheless, the definition follows the standard definition of the Lie bracket, which is why we call the defined operator also the Lie bracket.

Consider the vector fields f and g in \mathcal{E}^* defined as

$$f = \sum_{i=1}^n \tilde{f}_i \frac{\partial}{\partial x_i} + \sum_{k=0}^K \sum_{j=1}^m \bar{f}_{jk} \frac{\partial}{\partial u_j^{(k)}} + \sum_{l=1}^L \sum_{q=1}^m \hat{f}_{ql} \frac{\partial}{\partial z_q^{(-l)}} \quad (6)$$

and

$$g = \sum_{i=1}^n \tilde{g}_i \frac{\partial}{\partial x_i} + \sum_{k=0}^K \sum_{j=1}^m \bar{g}_{jk} \frac{\partial}{\partial u_j^{(k)}} + \sum_{l=1}^L \sum_{q=1}^m \hat{g}_{ql} \frac{\partial}{\partial z_q^{(-l)}}. \quad (7)$$

Let also $\xi = (x, u, \dots, u^{(K)}, z^{(-1)}, \dots, z^{(-L)})^T$.

Below we interpret the vector fields f and g as column vectors of their coefficients. Then the Lie bracket $[f, g]$ of the vector fields f and g can be defined in a standard way as

$$[f, g] := \frac{\partial f}{\partial \xi} g - \frac{\partial g}{\partial \xi} f.$$

Note that one can take the Lie bracket of vector fields of the same (finite) dimension. However, one can always represent the vector fields f and g in the form (6) and (7), respectively, by adding zero elements to proper positions. Unfortunately, the definition of the Lie bracket above does not give insight on how the elements of $[f, g]$ are computed component-wise. Such formulas would give a more thorough overview on how certain directions change when applying the Lie bracket. Thus, let

$$[f, g] = \sum_{i=1}^n \tilde{c}_i \frac{\partial}{\partial x_i} + \sum_{k=0}^K \sum_{j=1}^m \bar{c}_{jk} \frac{\partial}{\partial u_j^{(k)}} + \sum_{l=1}^L \sum_{q=1}^m \hat{c}_{ql} \frac{\partial}{\partial z_q^{(-l)}}.$$

Then, from the definition of the Lie bracket one has

$$\begin{aligned} \tilde{c}_i &= \sum_{\mu=1}^n \left(\frac{\partial \tilde{f}_i}{\partial x_\mu} \tilde{g}_\mu - \frac{\partial \tilde{g}_i}{\partial x_\mu} \tilde{f}_\mu \right) + \sum_{k=0}^K \sum_{j=1}^m \left(\frac{\partial \tilde{f}_i}{\partial u_j^{(k)}} \bar{g}_{jk} - \frac{\partial \tilde{g}_i}{\partial u_j^{(k)}} \bar{f}_{jk} \right) + \sum_{\ell=1}^L \sum_{q=1}^m \left(\frac{\partial \tilde{f}_i}{\partial z_q^{(-\ell)}} \hat{g}_{q\ell} - \frac{\partial \tilde{g}_i}{\partial z_q^{(-\ell)}} \hat{f}_{q\ell} \right), \\ \bar{c}_{jk} &= \sum_{i=1}^n \left(\frac{\partial \bar{f}_{jk}}{\partial x_i} \tilde{g}_i - \frac{\partial \bar{g}_{jk}}{\partial x_i} \tilde{f}_i \right) + \sum_{\mu=0}^K \sum_{\lambda=1}^m \left(\frac{\partial \bar{f}_{jk}}{\partial u_\lambda^{(\mu)}} \bar{g}_{\lambda\mu} - \frac{\partial \bar{g}_{jk}}{\partial u_\lambda^{(\mu)}} \bar{f}_{\lambda\mu} \right) + \sum_{\ell=1}^L \sum_{q=1}^m \left(\frac{\partial \bar{f}_{jk}}{\partial z_q^{(-\ell)}} \hat{g}_{q\ell} - \frac{\partial \bar{g}_{jk}}{\partial z_q^{(-\ell)}} \hat{f}_{q\ell} \right), \\ \hat{c}_{ql} &= \sum_{i=1}^n \left(\frac{\partial \hat{f}_{ql}}{\partial x_i} \tilde{g}_i - \frac{\partial \hat{g}_{ql}}{\partial x_i} \tilde{f}_i \right) + \sum_{k=0}^K \sum_{j=1}^m \left(\frac{\partial \hat{f}_{ql}}{\partial u_j^{(k)}} \bar{g}_{jk} - \frac{\partial \hat{g}_{ql}}{\partial u_j^{(k)}} \bar{f}_{jk} \right) + \sum_{\ell=1}^L \sum_{\mu=1}^m \left(\frac{\partial \hat{f}_{ql}}{\partial z_\mu^{(-\ell)}} \hat{g}_{\mu\ell} - \frac{\partial \hat{g}_{ql}}{\partial z_\mu^{(-\ell)}} \hat{f}_{\mu\ell} \right). \end{aligned}$$

Proposition 3. *The Lie bracket of the vector fields f and g has the following properties:*

- (i) $[f, f] = 0$,
- (ii) $[f + g, h] = [f, h] + [g, h]$,
- (iii) $[f, g + h] = [f, g] + [f, h]$,
- (iv) $[af, g] = a[f, g] + \langle da, g \rangle f$ for all $a \in \mathcal{H}$,
- (v) $[f, bg] = b[f, g] - \langle db, f \rangle g$ for all $b \in \mathcal{H}$,
- (vi) $[f, g]^p = [f^p, g^p]$,

(vii) $[f, g]^\sigma = [f^\sigma, g^\sigma]$.

Proof: The proofs of properties (i)–(v) are not different from the classical case in the n -dimensional space, see, for example, [3,15,18].

(vi) If the vector fields f and g are given by (6) and (7), respectively, then, by Proposition 1, one has

$$f^\rho = \sum_{i=1}^n \langle d\Phi_i, f \rangle^\rho \frac{\partial}{\partial x_i} + \sum_{k=0}^K \sum_{j=1}^m \langle du_j^{(k+1)}, f \rangle^\rho \frac{\partial}{\partial u_j^{(k)}} + \sum_{l=1}^L \sum_{q=1}^m \langle dz_q^{(-l+1)}, f \rangle^\rho \frac{\partial}{\partial z_q^{(-l)}}, \tag{8}$$

$$g^\rho = \sum_{i=1}^n \langle d\Phi_i, g \rangle^\rho \frac{\partial}{\partial x_i} + \sum_{k=0}^K \sum_{j=1}^m \langle du_j^{(k+1)}, g \rangle^\rho \frac{\partial}{\partial u_j^{(k)}} + \sum_{l=1}^L \sum_{q=1}^m \langle dz_q^{(-l+1)}, g \rangle^\rho \frac{\partial}{\partial z_q^{(-l)}}. \tag{9}$$

The vector field $[f^\rho, g^\rho]$ can be written as

$$[f^\rho, g^\rho] = \frac{\partial f^\rho}{\partial \xi} g^\rho - \frac{\partial g^\rho}{\partial \xi} f^\rho = \sum_i (\langle d(f^\rho)_i, g^\rho \rangle - \langle d(g^\rho)_i, f^\rho \rangle) \frac{\partial}{\partial \xi_i}, \tag{10}$$

where $(f^\rho)_i$ is the i th component of the vector f^ρ and, respectively, $(g^\rho)_i$ is that of the vector g^ρ . Considering (8), (9) and the property $\langle \omega^\rho, f^\rho \rangle = \langle \omega, f \rangle^\rho$ for all $\omega \in \mathcal{E}$ and $f \in \mathcal{E}^*$, (10) becomes

$$\begin{aligned} [f^\rho, g^\rho] &= \sum_{i=1}^n (\langle d\langle d\Phi_i, f \rangle, g \rangle^\rho - \langle d\langle d\Phi_i, g \rangle, f \rangle^\rho) \frac{\partial}{\partial x_i} \\ &+ \sum_{k=0}^K \sum_{j=1}^m (\langle d\langle du_j^{(k+1)}, f \rangle, g \rangle^\rho - \langle d\langle du_j^{(k+1)}, g \rangle, f \rangle^\rho) \frac{\partial}{\partial u_j^{(k)}} \\ &+ \sum_{l=1}^L \sum_{q=1}^m (\langle d\langle dz_q^{(-l+1)}, f \rangle, g \rangle^\rho - \langle d\langle dz_q^{(-l+1)}, g \rangle, f \rangle^\rho) \frac{\partial}{\partial z_q^{(-l)}}. \end{aligned}$$

Compute

$$\begin{aligned} \langle d\Phi_i, f \rangle &= \sum_{\mu=1}^n \frac{\partial \Phi_i}{\partial x_\mu} \tilde{f}_\mu + \sum_{j=1}^m \frac{\partial \Phi_i}{\partial u_j} \tilde{f}_{j0}, \\ \langle du_j^{(k+1)}, f \rangle &= \tilde{f}_{j(k+1)}, \quad k \geq 0, \\ \langle dz_q, f \rangle &= \sum_{\mu=1}^n \frac{\partial \chi_q}{\partial x_\mu} \tilde{f}_\mu + \sum_{j=1}^m \frac{\partial \chi_q}{\partial u_j} \tilde{f}_{j0}, \\ \langle dz_q^{(-l+1)}, f \rangle &= \hat{f}_{q(l-1)}, \quad l \geq 2 \end{aligned}$$

and

$$\begin{aligned} d\langle d\Phi_i, f \rangle &= \sum_{\mu=1}^n \left(d\left(\frac{\partial \Phi_i}{\partial x_\mu}\right) \tilde{f}_\mu + \frac{\partial \Phi_i}{\partial x_\mu} d\tilde{f}_\mu \right) + \sum_{j=1}^m \left(d\left(\frac{\partial \Phi_i}{\partial u_j}\right) \tilde{f}_{j0} + \frac{\partial \Phi_i}{\partial u_j} d\tilde{f}_{j0} \right), \\ d\langle du_j^{(k+1)}, f \rangle &= d\tilde{f}_{j(k+1)}, \quad k \geq 0, \\ d\langle dz_q, f \rangle &= \sum_{\mu=1}^n \left(d\left(\frac{\partial \chi_q}{\partial x_\mu}\right) \tilde{f}_\mu + \frac{\partial \chi_q}{\partial x_\mu} d\tilde{f}_\mu \right) + \sum_{j=1}^m \left(d\left(\frac{\partial \chi_q}{\partial u_j}\right) \tilde{f}_{j0} + \frac{\partial \chi_q}{\partial u_j} d\tilde{f}_{j0} \right), \\ d\langle dz_q^{(-l+1)}, f \rangle &= d\hat{f}_{q(l-1)}, \quad l \geq 2. \end{aligned}$$

Therefore,

$$\begin{aligned}
\langle d \langle d\Phi_i, f \rangle, g \rangle^\rho &= \sum_{\mu=1}^n \left(\tilde{f}_\mu^\rho \left\langle d \left(\frac{\partial \Phi_i}{\partial x_\mu} \right), g \right\rangle^\rho + \left(\frac{\partial \Phi_i}{\partial x_\mu} \right)^\rho \langle d\tilde{f}_\mu, g \rangle^\rho \right) \\
&+ \sum_{j=1}^m \left(\tilde{f}_{j0}^\rho \left\langle d \left(\frac{\partial \Phi_i}{\partial u_j} \right), g \right\rangle^\rho + \left(\frac{\partial \Phi_i}{\partial u_j} \right)^\rho \langle d\tilde{f}_{j0}, g \rangle^\rho \right) \\
&= \sum_{\mu=1}^n \left(\sum_{\lambda=1}^n \left(\frac{\partial^2 \Phi_i}{\partial x_\mu \partial x_\lambda} \right)^\rho \tilde{f}_\mu^\rho \tilde{g}_\lambda^\rho + \sum_{r=1}^m \left(\frac{\partial^2 \Phi_i}{\partial x_\mu \partial u_r} \right)^\rho \tilde{f}_\mu^\rho \tilde{f}_{r0}^\rho + \left(\frac{\partial \Phi_i}{\partial x_\mu} \right)^\rho \langle d\tilde{f}_\mu, g \rangle^\rho \right) \\
&+ \sum_{j=1}^m \left(\sum_{\lambda=1}^n \left(\frac{\partial^2 \Phi_i}{\partial u_j \partial x_\lambda} \right)^\rho \tilde{f}_{j0}^\rho \tilde{g}_\lambda^\rho + \sum_{r=1}^m \left(\frac{\partial^2 \Phi_i}{\partial u_j \partial u_r} \right)^\rho \tilde{f}_{j0}^\rho \tilde{f}_{r0}^\rho + \left(\frac{\partial \Phi_i}{\partial u_j} \right)^\rho \langle d\tilde{f}_{j0}, g \rangle^\rho \right), \\
\langle d \langle d\mathbf{u}_j^{(k+1)}, f \rangle, g \rangle^\rho &= \langle d\tilde{f}_{j(k+1)}, g \rangle^\rho, \quad k \geq 0, \\
\langle d \langle dz_q, f \rangle, g \rangle^\rho &= \sum_{\mu=1}^n \left(\tilde{f}_\mu^\rho \left\langle d \left(\frac{\partial \chi_q}{\partial x_\mu} \right), g \right\rangle^\rho + \left(\frac{\partial \chi_q}{\partial x_\mu} \right)^\rho \langle d\tilde{f}_\mu, g \rangle^\rho \right) \\
&+ \sum_{j=1}^m \left(\tilde{f}_{j0}^\rho \left\langle d \left(\frac{\partial \chi_q}{\partial u_j} \right), g \right\rangle^\rho + \left(\frac{\partial \chi_q}{\partial u_j} \right)^\rho \langle d\tilde{f}_{j0}, g \rangle^\rho \right) \\
&= \sum_{\mu=1}^n \left(\sum_{\lambda=1}^n \left(\frac{\partial^2 \chi_q}{\partial x_\mu \partial x_\lambda} \right)^\rho \tilde{f}_\mu^\rho \tilde{g}_\lambda^\rho + \sum_{r=1}^m \left(\frac{\partial^2 \chi_q}{\partial x_\mu \partial u_r} \right)^\rho \tilde{f}_\mu^\rho \tilde{f}_{r0}^\rho + \left(\frac{\partial \chi_q}{\partial x_\mu} \right)^\rho \langle d\tilde{f}_\mu, g \rangle^\rho \right) \\
&+ \sum_{j=1}^m \left(\sum_{\lambda=1}^n \left(\frac{\partial^2 \chi_q}{\partial u_j \partial x_\lambda} \right)^\rho \tilde{f}_{j0}^\rho \tilde{g}_\lambda^\rho + \sum_{r=1}^m \left(\frac{\partial^2 \chi_q}{\partial u_j \partial u_r} \right)^\rho \tilde{f}_{j0}^\rho \tilde{f}_{r0}^\rho + \left(\frac{\partial \chi_q}{\partial u_j} \right)^\rho \langle d\tilde{f}_{j0}, g \rangle^\rho \right), \\
\langle d \langle dz_q^{(-l+1)}, f \rangle, g \rangle^\rho &= \langle d\hat{f}_{q(l-1)}, g \rangle^\rho, \quad l \geq 2.
\end{aligned}$$

By changing f and g in the latter expressions one obtains similar expressions for $\langle d \langle d\Phi_i, g \rangle, f \rangle^\rho$, $\langle d \langle dz_q, g \rangle, f \rangle^\rho$, $\langle d \langle d\mathbf{u}_j^{(k+1)}, g \rangle, f \rangle^\rho$ and $\langle d \langle dz_q^{(-l+1)}, g \rangle, f \rangle^\rho$. Since one deals with continuous functions, then one can switch the order of partial derivatives, which yields

$$\begin{aligned}
[f^\rho, g^\rho] &= \sum_{i=1}^n \left(\sum_{\mu=1}^n \left(\frac{\partial \Phi_i}{\partial x_\mu} \right)^\rho (\langle d\tilde{f}_\mu, g \rangle^\rho - \langle d\tilde{g}_\mu, f \rangle^\rho) + \sum_{r=1}^m \left(\frac{\partial \Phi_i}{\partial u_r} \right)^\rho (\langle d\tilde{f}_{r0}, g \rangle^\rho - \langle d\tilde{g}_{r0}, f \rangle^\rho) \right) \frac{\partial}{\partial x_i} \\
&+ \sum_{k=0}^K \sum_{j=1}^m (\langle d\tilde{f}_{j(k+1)}, g \rangle^\rho - \langle d\tilde{g}_{j(k+1)}, f \rangle^\rho) \frac{\partial}{\partial u_j^{(k)}} \quad (11) \\
&+ \sum_{s=1}^m \left(\sum_{\mu=1}^n \left(\frac{\partial \chi_s}{\partial x_\mu} \right)^\rho (\langle d\tilde{f}_\mu, g \rangle^\rho - \langle d\tilde{g}_\mu, f \rangle^\rho) + \sum_{r=1}^m \left(\frac{\partial \chi_s}{\partial u_r} \right)^\rho (\langle d\tilde{f}_{r0}, g \rangle^\rho - \langle d\tilde{g}_{r0}, f \rangle^\rho) \right) \frac{\partial}{\partial z_s^{(-1)}} \\
&+ \sum_{l=2}^L \sum_{q=1}^m (\langle d\hat{f}_{q(l-1)}, g \rangle^\rho - \langle d\hat{g}_{q(l-1)}, f \rangle^\rho) \frac{\partial}{\partial z_q^{(-l)}}.
\end{aligned}$$

Next, by Proposition 1 one has

$$[f, g]^\rho = \sum_{i=1}^n \langle d\Phi_i, [f, g] \rangle^\rho \frac{\partial}{\partial x_i} + \sum_{k=0}^K \sum_{j=1}^m \langle d\mathbf{u}_j^{(k+1)}, [f, g] \rangle^\rho \frac{\partial}{\partial u_j^{(k)}} + \sum_{l=1}^L \sum_{q=1}^m \langle dz_q^{(-l+1)}, [f, g] \rangle^\rho \frac{\partial}{\partial z_q^{(-l)}}. \quad (12)$$

Also, one can write

$$[f, g] = \frac{\partial f}{\partial \xi} g - \frac{\partial g}{\partial \xi} f = \sum_i (\langle df_i, g \rangle - \langle dg_i, f \rangle) \frac{\partial}{\partial \xi_i}, \quad (13)$$

where f_i is the i th element of the vector f and, respectively, g_i is that of g . Using (13) to compute the right-hand side of (12) one gets that $[f, g]^p$ is equal to the right-hand side of (11), i.e., $[f, g]^p = [f^p, g^p]$.

(vii) By property (vi) one has $[f^\sigma, g^\sigma]^p = [f, g] = ([f, g]^\sigma)^p$, which yields (vii). \square

3.1. Involutivity of vector spaces of vector fields

Consider a vector space of vector fields $\Delta = \text{span}_{\mathcal{H}}\{f^1, \dots, f^k\}$ for some $f^i \in \mathcal{E}^*$, $i = 1, \dots, k$.

Definition 3. A vector space Δ is said to be involutive if $[f, g] \in \Delta$ for all $f, g \in \Delta$.

The condition in Definition 3 requires, in principle, infinite number of computations, since a vector space Δ contains infinite number of vector fields. However, one can show that the condition in Definition 3 must be checked only for the basis vector fields f^i , $i = 1, \dots, k$. Note that the latter is not so trivial, since, by (iv) and (v) of Proposition 3, the Lie bracket is not a \mathcal{H} -bilinear map.

Proposition 4. A vector space $\Delta = \text{span}_{\mathcal{H}}\{f^1, \dots, f^k\}$ is involutive if and only if $[f^i, f^j] \in \Delta$ for $i, j = 1, \dots, k$.

Proof: The proof is similar to those in [15,18] for a classical case. \square

Involutivity is a key property in differential geometric control. In particular it allows to define different system transformations, discussed in the next section. In a classical differential geometric approach any involutive vector space of vector fields in $\text{span}\{\partial/\partial x_1, \dots, \partial/\partial x_n\}$ corresponds to an integrable vector space of 1-forms \mathcal{A} in $\text{span}\{dx_1, \dots, dx_n\}$. Here, however, involutivity is only a necessary condition for the existence of such integrable vector space of 1-forms. This happens because the functions in \mathcal{H} can also depend on variables $u_i^{(j)}$ and $z_i^{(-j-1)}$ for $i = 1, \dots, m$, $j \geq 0$. Therefore, additional conditions, (ii) and (iii) below, have to be checked.

Proposition 5. Consider a vector space of vector fields $\Delta = \text{span}_{\mathcal{H}}\{f^1, \dots, f^k\} \subseteq \text{span}_{\mathcal{H}}\{\partial/\partial x_1, \dots, \partial/\partial x_n\}$. Then there exists an integrable $(n-k)$ -dimensional vector space of 1-forms $\mathcal{A} \subseteq \text{span}_{\mathcal{H}}\{dx_1, \dots, dx_n\}$, such that $\langle \mathcal{A}, \Delta \rangle \equiv 0$ if and only if

(i) Δ is involutive,

(ii) $[f, \partial/\partial u_i^{(j)}] \in \Delta$ for all $f \in \Delta$, $i = 1, \dots, m$ and $j = 0, \dots, K$,

(iii) $[f, \partial/\partial z_i^{(-j)}] \in \Delta$ for all $f \in \Delta$, $i = 1, \dots, m$ and $j = 1, \dots, L$,

where K (L) is the highest (lowest) shift of u (z) present in the coefficients of the basis elements of Δ .

Proof: Necessity. Existence of an integrable $(n-k)$ -dimensional vector space of 1-forms $\mathcal{A} \subseteq \text{span}_{\mathcal{H}}\{dx_1, \dots, dx_n\}$, such that $\langle \mathcal{A}, \Delta \rangle \equiv 0$ yields that the vector space of vector fields

$$\Delta + \text{span}_{\mathcal{H}}\{\partial/\partial u_1, \dots, \partial/\partial u_1^{(K)}, \dots, \partial/\partial u_m^{(K)}, \partial/\partial z_1^{(-1)}, \dots, \partial/\partial z_m^{(-L)}\}$$

is involutive. Thus, the conditions (i), (ii) and (iii) must be true.

Sufficiency. The conditions (i), (ii) and (iii) yield that the vector space of vector fields

$$\Delta + \text{span}_{\mathcal{H}}\{\partial/\partial u_1, \dots, \partial/\partial u_1^{(K)}, \dots, \partial/\partial u_m^{(K)}, \partial/\partial z_1^{(-1)}, \dots, \partial/\partial z_m^{(-L)}\}$$

is involutive. Thus, there must exist an integrable $(n-k)$ -dimensional vector space of 1-forms $\mathcal{A} \subseteq \text{span}_{\mathcal{H}}\{dx_1, \dots, dx_n\}$, such that $\langle \mathcal{A}, \Delta \rangle \equiv 0$. \square

4. SYSTEM TRANSFORMATIONS

In this section we investigate how different system transformations affect a given vector field, i.e., how a vector field transforms under different system transformations. All the considered transformations are in the form

$$\begin{aligned}\eta &= \Psi_x(x, u, \dots, u^{(S)}, z^{(-1)}, \dots, z^{(-R)}), \\ v^{(k)} &= \Psi_{uk}(x, u, \dots, u^{(k+\tilde{n}-n)}), \\ w^{(-l)} &= \Psi_{zl}(x, z^{(-1)}, \dots, z^{(-l)}),\end{aligned}\quad (14)$$

where $\eta \in \mathbb{R}^{\tilde{n}}$, $\tilde{n} \geq n$, is a new state vector, v is a new input vector and w corresponds to a different choice of function χ , for $S, R \in \mathbb{N}$, $k \geq 0$, $l \geq 1$. Let $\Psi = (\Psi_x^T, \Psi_{uk}^T, \Psi_{zl}^T)^T$. We also assume, that the transformation (14) is invertible, meaning that there exists Ψ^{-1} . Transformations of the form (14) include the typical transformations in nonlinear control, for example:

- A state transformation, in which case one has $\tilde{n} = n$ and

$$\begin{aligned}\eta &= \Psi_x(x) \\ v^{(k)} &= u^{(k)} \\ w^{(-l)} &= z^{(-l)}.\end{aligned}\quad (15)$$

- A parametrized state transformation, in which case one has $\tilde{n} = n$ and

$$\begin{aligned}\eta &= \Psi_x(x, u, \dots, u^{(S)}) \\ v^{(k)} &= u^{(k)} \\ w^{(-l)} &= z^{(-l)}.\end{aligned}\quad (16)$$

- A regular static state feedback, in which case one has $\tilde{n} = n$ and

$$\begin{aligned}\eta &= x \\ v^{(k)} &= \Psi_{uk}(x, u, \dots, u^{(k)}) \\ w^{(-l)} &= z^{(-l)}.\end{aligned}\quad (17)$$

- A regular dynamic state feedback, in which case one has

$$\begin{aligned}\eta_i &= x_i & i = 1, \dots, n \\ \eta_i &= \Psi_{xi}(x, u, \dots, u^{(S)}) & i = n+1, \dots, \tilde{n} \\ v^{(k)} &= \Psi_{uk}(x, u, \dots, u^{(k+\tilde{n}-n)}) \\ w^{(-l)} &= z^{(-l)}.\end{aligned}\quad (18)$$

- Selection of different function χ for the definition of the z variable, in which case one has

$$\begin{aligned}\eta &= x \\ v^{(k)} &= u^{(k)} \\ w^{(-l)} &= \Psi_{zl}(x, z^{(-1)}, \dots, z^{(-l)}).\end{aligned}\quad (19)$$

Consider the vector field (3) and denote by $\Psi(f)$ the vector field f after the coordinate transformation (14). We define the transformed vector field $\Psi(f)$ in a similar manner as it is defined in the standard differential geometric case (see Eq. (2.85) in [15]) for a state transformation. Namely, if one interprets

the vector fields f and $\Psi(f)$ as column vectors of their coefficients, then $\Psi(f) = \left(\frac{\partial \Psi}{\partial \xi} f\right) \circ \Psi^{-1}$, where $\xi = (x, u, \dots, u^{(K)}, z^{(-1)}, \dots, z^{(-L)})^T$. Let us also define $\Psi(f)$ component-wise. If

$$\Psi(f) = \sum_{i=1}^{\tilde{n}} \tilde{F}_i \frac{\partial}{\partial \eta_i} + \sum_{k=0}^{\max\{0, K\}} \sum_{j=1}^m \tilde{F}_{jk} \frac{\partial}{\partial v_j^{(k)}} + \sum_{l=1}^{\max\{1, L\}} \sum_{q=1}^m \hat{F}_{ql} \frac{\partial}{\partial w_q^{(-l)}}, \quad (20)$$

then the coefficients are computed as

$$\begin{aligned} \tilde{F}_i &:= \langle d\eta_i, \Psi(f) \rangle = \langle d\Psi_{xi}, f \rangle \circ \Psi^{-1} \\ &= \sum_{\mu=1}^n \left(\frac{\partial \Psi_{xi}}{\partial x_\mu} \tilde{f}_\mu \right) \circ \Psi^{-1} + \sum_{r \geq 0, s=1}^m \left(\left(\frac{\partial \Psi_{xi}}{\partial u_s^{(r)}} \tilde{f}_{sr} \right) \circ \Psi^{-1} + \left(\frac{\partial \Psi_{xi}}{\partial z_s^{(-r)}} \hat{f}_{sr} \right) \circ \Psi^{-1} \right), \\ \tilde{F}_{jk} &:= \langle dv_j^{(k)}, \Psi(f) \rangle = \langle d\Psi_{ukj}, f \rangle \circ \Psi^{-1} = \sum_{\mu=1}^n \left(\frac{\partial \Psi_{ukj}}{\partial x_\mu} \tilde{f}_\mu \right) \circ \Psi^{-1} + \sum_{r \geq 0, s=1}^m \left(\frac{\partial \Psi_{ukj}}{\partial u_s^{(r)}} \tilde{f}_{sr} \right) \circ \Psi^{-1}, \quad (21) \\ \hat{F}_{ql} &:= \langle dw_q^{(-l)}, \Psi(f) \rangle = \langle d\Psi_{z_lq}, f \rangle \circ \Psi^{-1} = \sum_{\mu=1}^n \left(\frac{\partial \Psi_{z_lq}}{\partial x_\mu} \tilde{f}_\mu \right) \circ \Psi^{-1} + \sum_{r \geq 1, s=1}^m \left(\frac{\partial \Psi_{z_lq}}{\partial z_s^{(-r)}} \hat{f}_{sr} \right) \circ \Psi^{-1}. \end{aligned}$$

Proposition 6. System transformations have the following properties:

- (i) $\Psi(f + g) = \Psi(f) + \Psi(g)$,
- (ii) $\Psi(f^\rho) = (\Psi(f))^\rho$,
- (iii) $\Psi(f^\sigma) = (\Psi(f))^\sigma$,
- (iv) $[\Psi(f), \Psi(g)] = \Psi([f, g])$.

Proof: For simplification let $\xi = (x, u, \dots, u^{(k)}, z^{(-1)}, \dots, z^{(-l)})^T$ and $\tilde{\xi} = (\eta, v, \dots, v^{(k)}, w^{(-1)}, \dots, w^{(-l)})^T$ for $k \geq 0$ and $l > 0$.

(i) Replace f by $f + g$ in (21). Then the computations of the coefficients of $\Psi(f + g)$ yield directly $\Psi(f + g) = \Psi(f) + \Psi(g)$.

(ii) By (21) one has that the coefficients of $\Psi(f^\rho)$ are in the form $\langle d\Psi_i, f^\rho \rangle \circ \Psi^{-1}$. By definition of the backward shift of a vector field $\langle \omega, f^\rho \rangle = \langle \omega^\sigma, f \rangle^\rho$ for all $\omega \in \mathcal{E}$. Thus, one must have $\langle d\Psi_i, f^\rho \rangle \circ \Psi^{-1} = \langle d\Psi_i^\sigma, f \rangle^\rho \circ \Psi^{-1}$. Next, we use the fact that the transformation (14) commutes with the backward shift of a function in \mathcal{K} (this can be shown by direct computations). Since $\langle d\Psi_i^\sigma, f \rangle \in \mathcal{K}$ and the transformation (14) also commutes with the scalar product operator, then one can conclude that $\langle d\Psi_i, f^\rho \rangle \circ \Psi^{-1} = \langle d\Psi_i^\sigma, f \rangle^\rho \circ \Psi^{-1} = \left\langle d\tilde{\xi}_i^\sigma, \Psi(f) \right\rangle^\rho$. The latter is, by Proposition 1, the i th coefficient of $(\Psi(f))^\rho$.

(iii) By using the property (ii) one can compute $(\Psi(f^\sigma))^\rho = \Psi(f) = ((\Psi(f))^\sigma)^\rho$. Then obviously $\Psi(f^\sigma) = (\Psi(f))^\sigma$ must be true.

(iv) By considering the relation (13), straightforward computations show that

$$\begin{aligned} \Psi([f, g]) &= \sum_i \left\langle d\tilde{\xi}_i, \Psi([f, g]) \right\rangle \frac{\partial}{\partial \tilde{\xi}_i} = \sum_i \langle d\Psi_i, [f, g] \rangle \circ \Psi^{-1} \frac{\partial}{\partial \tilde{\xi}_i} \\ &= \sum_i \sum_j \left(\frac{\partial \Psi_i}{\partial \xi_j} (\langle df_j, g \rangle - \langle dg_j, f \rangle) \right) \circ \Psi^{-1} \frac{\partial}{\partial \tilde{\xi}_i}, \quad (22) \end{aligned}$$

where f_j is the j th element of the vector f and, respectively, g_j is that of g . On the other hand, by (21),

$$\begin{aligned} \Psi(f) &= \sum_i \left\langle d\tilde{\xi}_i, \Psi(f) \right\rangle \frac{\partial}{\partial \tilde{\xi}_i} = \sum_i \langle d\Psi_i, f \rangle \circ \Psi^{-1} \frac{\partial}{\partial \tilde{\xi}_i}, \\ \Psi(g) &= \sum_i \left\langle d\tilde{\xi}_i, \Psi(g) \right\rangle \frac{\partial}{\partial \tilde{\xi}_i} = \sum_i \langle d\Psi_i, g \rangle \circ \Psi^{-1} \frac{\partial}{\partial \tilde{\xi}_i}. \end{aligned}$$

Then, by using (13) again one has

$$\begin{aligned} [\Psi(f), \Psi(g)] &= \sum_i \left(\langle d \langle d \tilde{\xi}_i, \Psi(f) \rangle, \Psi(g) \rangle - \langle d \langle d \tilde{\xi}_i, \Psi(g) \rangle, \Psi(f) \rangle \right) \frac{\partial}{\partial \tilde{\xi}_i} \\ &= \sum_i \left(\langle d \langle d \Psi_i, f \rangle, g \rangle - \langle d \langle d \Psi_i, g \rangle, f \rangle \right) \circ \Psi^{-1} \frac{\partial}{\partial \tilde{\xi}_i}. \end{aligned} \quad (23)$$

Compute

$$\begin{aligned} \langle d \Psi_i, f \rangle &= \sum_j \frac{\partial \Psi_i}{\partial \xi_j} f_j, \\ d \langle d \Psi_i, f \rangle &= \sum_j \left(d \left(\frac{\partial \Psi_i}{\partial \xi_j} \right) f_j + \frac{\partial \Psi_i}{\partial \xi_j} d f_j \right), \\ \langle d \langle d \Psi_i, f \rangle, g \rangle &= \sum_j \left(f_j \left\langle d \left(\frac{\partial \Psi_i}{\partial \xi_j} \right), g \right\rangle + \frac{\partial \Psi_i}{\partial \xi_j} \langle d f_j, g \rangle \right), \\ &= \sum_j \left(\sum_{\mu} \frac{\partial^2 \Psi_i}{\partial \xi_j \partial \xi_{\mu}} f_j g_{\mu} + \frac{\partial \Psi_i}{\partial \xi_j} \langle d f_j, g \rangle \right). \end{aligned}$$

Similarly one can compute $\langle d \Psi_i, g \rangle$, $d \langle d \Psi_i, g \rangle$ and $\langle d \langle d \Psi_i, g \rangle, f \rangle$. Since all the functions are assumed to be continuous, then $\frac{\partial^2 \Psi_i}{\partial \xi_j \partial \xi_{\mu}} = \frac{\partial^2 \Psi_i}{\partial \xi_{\mu} \partial \xi_j}$ and thus (23) becomes

$$[\Psi(f), \Psi(g)] = \sum_i \sum_j \frac{\partial \Psi_i}{\partial \xi_j} \left(\langle d f_j, g \rangle - \langle d g_j, f \rangle \right) \circ \Psi^{-1} \frac{\partial}{\partial \tilde{\xi}_i}. \quad (24)$$

Thus, by (22), $\Psi([f, g]) = [\Psi(f), \Psi(g)]$. □

Proposition 6 yields the following result.

Corollary 1. *Selection of function χ for definition of the vector z does not affect applying the forward or backward shift operators to a vector field in a sense that there always exists a transformation of the form (19) that relates the forward or backward shifts of a vector field for different choices of function χ .*

Proof: Note that the different choices of function χ for the definition of the vector z are related by a transformation of the form (19). Thus, by (ii) or (iii) of Proposition 6 the same transformation of the form (19) relates the backward or forward shifts of a vector field for different choices of χ . □

5. CONCLUSIONS

It has been shown in the paper that recently defined forward and backward shifts of vector fields commute with both the Lie bracket operator and a certain system transformation on a vector field. These properties are necessary for studying various structural control problems. Note that the transformations considered in Section 4 include the transformation of the extended state map. More precisely, it was demonstrated that there always exists a transformation of the form (19) between two different choices of the function χ , resulting in two different extended state map variables z and w . Since, by Proposition 6, the transformation (19) commutes with the forward and backward shifts as well as with the Lie bracket, then we have shown in Corollary 1 that the choice of χ does not affect applying the above-mentioned operators to a vector field.

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Vektorväljade edasi- ja tagasihete omadustest

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Töös uuritakse vektorväljade mõningaid hiljuti defineeritud edasi- ja tagasihete omadusi. Artikli peamine eesmärk on näidata, et vektorväljade edasi- ja tagasiheteked commuteeruvad vektorväljade Lie sulgude operaatoriga ning samuti mõningate mittelineaarses juhtimisteoorias enamkasutatavate muutujate teisen- dustega. Viimaste hulka kuuluvad näiteks klassikaline ja parametrizeeritud olekuteisendus ning staatiline ja dünaamiline tagasihete. Töös tõestatud tulemused on olulised edasises uurimustöös lahendamaks selle meetodikaga erinevaid juhtimisprobleeme, kus rakendatakse eelmainitud operaatoreid ning teisen- dusi.