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ALGEBRAIC
HYPERSTRUCTURES

δ-r-hyperideals and ϕ-δ-r-hyperideals of commutative Krasner hyperrings

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Abstract. This paper deals with an important class of multialgebras, called Krasner hyperrings. Our purpose is to define the expansion of r-hyperideals and to extend this concept to ϕ-δ-r-hyperideal in commutative Krasner hyperrings with nonzero identity. δ-r-hyperideals of commutative Krasner hyperrings are studied. Some properties of ϕ-δ-r-hyperideals are investigated and several examples are provided.

Keywords: r-hyperideal, δ-r-hyperideal, ϕ-δ-primary hyperideal, ϕ-δ-r-hyperideal.

1. INTRODUCTION

Several authors have been studying prime and primary ideals that are quite important in commutative rings. In 2001, Zhao [13] introduced the concept of expansion of ideals of a commutative ring. An expansion of ideals is a function δ that assigns to each ideal $\mathscr{N}$ of $\mathfrak{R}$ another ideal $\delta(\mathscr{N})$ of the same ring $\mathfrak{R}$ such that for all ideals $\mathscr{N}$ of $\mathfrak{R}$ $\mathscr{N} \subseteq \delta(\mathscr{N})$ and if $\mathscr{N} \subseteq \mathscr{M}$ where $\mathscr{N}$ and $\mathscr{M}$ are ideals of $\mathfrak{R}$, then $\delta(\mathscr{N}) \subseteq \delta(\mathscr{M})$. Let δ be an ideal expansion. An ideal $\mathscr{N}$ of $\mathfrak{R}$ is called a δ-primary ideal of $\mathfrak{R}$, if $ab \in \mathscr{N}$, then $a \in \mathscr{N}$ or $b \in \delta(\mathscr{N})$, for all $a, b \in \mathfrak{R}$.

Furthermore, the concepts of ϕ-prime and ϕ-primary ideals are examined in [4,7]. A reduction of ideals is a function ϕ that maps each ideal $\mathscr{N}$ of $\mathfrak{R}$ to another ideal $\phi(\mathscr{N})$ of the same ring $\mathfrak{R}$ satisfying the following terms: i) for all ideals $\mathscr{N}$ of $\mathfrak{R}$, $\phi(\mathscr{N}) \subseteq \mathscr{N}$, ii) if $\mathscr{N} \subseteq \mathscr{M}$ with $\mathscr{N}$ and $\mathscr{M}$ being ideals of $\mathfrak{R}$, then $\phi(\mathscr{N}) \subseteq \phi(\mathscr{M})$. Suppose that ϕ is an ideal reduction. A proper ideal $\mathscr{N}$ of $\mathfrak{R}$ is called a ϕ-prime ideal of $\mathfrak{R}$, if $ab \in \mathscr{N} - \phi(\mathscr{N})$, then $a \in \mathscr{N}$ or $b \in \mathscr{N}$, for all $a, b \in \mathfrak{R}$. $\mathscr{N}$ is called a ϕ-primary ideal of $\mathfrak{R}$, if $ab \in \mathscr{N} - \phi(\mathscr{N})$, then $a \in \mathscr{N}$ or $b \in \sqrt{\mathscr{N}}$, for all $a, b \in \mathfrak{R}$. Later, Jaber [16] characterized ϕ-δ-primary ideals in commutative rings. Assume that δ is an ideal expansion and ϕ is an ideal reduction. An ideal $\mathscr{N}$ of $\mathfrak{R}$ is called a ϕ-δ- primary ideal of $\mathfrak{R}$, if $ab \in \mathscr{N} - \phi(\mathscr{N})$, then $a \in \mathscr{N}$ or $b \in \delta(\mathscr{N})$, for all $a, b \in \mathfrak{R}$.

Recently, Mohamadian [24] investigated the properties of the class of r-ideals. A proper ideal $\mathscr{N}$ in a commutative ring $\mathfrak{R}$ is called an r-ideal (resp., pr-ideal), if $ab \in \mathscr{N}$ with $ann(a) = 0$ implies that $b \in \mathscr{N}$

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(resp., $b^n \in \mathscr{N}$, for some $n \in \mathbb{N}$), for each $a, b \in \mathfrak{R}$. Ugurlu [29] studied generalizations of r-ideals. Assume that $\mathfrak{R}$ is a commutative ring, $\mathscr{N}$ is a proper ideal of $\mathfrak{R}$ and $\phi : Id(\mathfrak{R}) \to Id(\mathfrak{R}) \cup \{\emptyset\}$ is a function. $\mathscr{N}$ is called a ϕ-r-ideal (resp., ϕ-pr-ideal) if $ab \in \mathscr{N} - \phi(\mathscr{N})$ with $ann(a) = 0$ implies that $b \in \mathscr{N}$ (resp., $b^n \in \mathscr{N}$, for some $n \in \mathbb{N}$), for $a, b \in \mathfrak{R}$.

Hyperstructures were introduced at the 8th Congress of Scandinavian Mathematicians in 1934 by a French mathematician Marty [19]. In the sense of Marty, let $G \neq \emptyset$, a mapping $\circ : G \times G \longrightarrow \mathscr{P}^*(G)$ is a hyperoperation and $(G, \circ)$ is hypergroupoid. Let A and B be two non-empty subsets of G and $x \in G$. Then we define $A \circ B = \bigcup\limits_{a\in A, b\in B} a \circ b$, $A \circ x = A \circ \{x\}$, and $x \circ B = \{x\} \circ B$. If $(G, \circ)$ is a hypergroupoid and $x \circ (y \circ z) = (x \circ y) \circ z$, for $\forall\, x, y, z \in G$, which means that $\bigcup\limits_{u\in x\circ y} u \circ z = \bigcup\limits_{v\in y\circ z} x \circ v$, then G is a semihypergroup. When $(G, \circ)$ is a semihypergroup with $x \circ G = G \circ x = G$, for $\forall x \in G$, then $(G, \circ)$ is called a hypergroup. A nonempty set G along with hyperoperation $+$ is called a canonical hypergroup if the following axioms hold: i) $x + (y + z) = (x + y) + z$, for $x, y, z \in G$; ii) $x + y = y + x$, for $x, y \in G$; iii) there exists $0 \in G$ such that $x + 0 = \{x\}$, for any $x \in G$; iv) for any $x \in G$, there exists a unique element $x' \in G$, such that $0 \in x + x'$ (x' is called the opposite of x and is denoted by $-x$). v) $z \in x + y$ implies that $y \in -x + z$ and $x \in z - y$, that is $(G, +)$ is reversible [22]. A comprehensive review of the theory of hypergroups can be found in [5,8,20].

After that, many papers and books concerning hyperstructure theory have been published. Corsini [5] described the results on hypergroups and furthermore, in [6], Corsini and Leoreanu illustrated some of the most recent and interesting applications, that is, those to geometry, graphs and hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras and C-algebras, artificial intelligence, and probability. Davvaz and Leoreanu [9] wrote another important monograph on algebraic hyperstructures, which contains the main results obtained in hyperring theory till the publication of it, but also an outline of applications of hyperstructures. Recently, Davvaz and Leoreanu-Fotea [11] published a monograph which presents an updated study of hypergroups dedicated both to the theoretical part and to the applications. It was Davvaz [8] who wrote another excellent monograph on polygroups as a certain subclass of hypergroups used to study color algebra and combinatorics. Another well-known author of hyperstructures, T. Vougiouklis [30] presented a monograph which deals above all with generalizations of classical algebraic hyperstructures, the so-called H_v-structures and their applications. In [10] Davvaz and Vougiouklis continued the work by exploring weak hyperstructures with natural examples and applications in natural sciences. These books form the basis of knowledge on hyperstructures, contain an updated complete bibliography on hyperstructures and related concepts, and are strongly recommended for mathematicians and researchers in the field. We refer to these sources for several concepts used in our work.

Although there are different types of hyperrings, in this paper we consider Krasner hyperrings [17], which is one of the most studied class of hyperrings. If $(\mathfrak{R}, +)$ is a canonical hypergroup, $(\mathfrak{R}, \cdot)$ is a semigroup that has 0 as a bilaterally absorbing element for all elements of $\mathfrak{R}$ and $\cdot$ is distributive with respect to $+$, then $(G, +, \cdot)$ is known as the Krasner hyperring (see [18]). After Krasner introduced the definition of hyperring, the study of hyperrings has been of great importance and has a multitude of applications to other disciplines, for example see [2,3,23]. Students of Krasner, namely J. Mittas and D. Stratigopoulos, well-known authors, have studied hyperrings and hyperfields. Other researchers including P. Corsini, C. Massouros, A. Nakassis, T. Vougiouklis, T. Koguentsof, A. Dramalidis, S. Spartalis, B. Davvaz, R. Ameri, V. Leoreanu-Fotea, I. Cristea, G. Pinotsis and Y. Kemprasit have also made notable contributions to this subject. Some fundamental notions and results in the theory of hyperrings can be found in [12,21,25,28,31].

Let $\mathfrak{R}$ be a hyperring. For $a \in \mathfrak{R}$, we define $ann(a) = \{r \in \mathfrak{R} : ra = 0\}$. If $ann(a) = 0$ (resp., $ann(a) \neq 0$), a is said to be regular (resp., zero divisor). We use the notion $r(\mathfrak{R})$ (resp., $zd(\mathfrak{R})$) to denote the set of all regular elements (resp., zero divisors) [1,15]. If $\mathscr{N}$ is a hyperideal of $\mathfrak{R}$ and A a subset of $\mathfrak{R}$, then we denote $(\mathscr{N} : A) = \{x \in \mathfrak{R}, x \cdot A \subseteq \mathscr{N}\}$. It is clear that $(0 : A) = ann(A)$.

According to Davvaz's definition, a proper hyperideal $\mathscr{N}$ in a commutative Krasner hyperring $\mathfrak{R}$ is called an r-hyperideal (resp., pr-hyperideal), if $a \cdot b \in \mathscr{N}$ with $ann(a) = 0$ implies that $b \in \mathscr{N}$ (resp.,

$b^n \in \mathscr{N}$, for some $n \in \mathbb{N}$), for each $a,b \in \mathfrak{R}$ [26]. Some other results on ϕ-δ-primary hyperideals and r-hyperideals in Krasner hyperrings have been obtained recently [14,32].

Assume that $\mathfrak{R}$ is a Krasner hyperring, $\phi : Id(\mathfrak{R}) \to Id(\mathfrak{R}) \cup \{\emptyset\}$ is a function and $\emptyset \neq \mathscr{N} \in Id(\mathfrak{R})$. Then $\mathscr{N}$ is said to be a ϕ-prime (resp. ϕ-primary) hyperideal of $\mathfrak{R}$ if whenever $r,s \in \mathfrak{R}$ and $r \cdot s \in \mathscr{N} - \phi(\mathscr{N})$, then $r \in \mathscr{N}$ or $s \in \mathscr{N}$ (resp., $r \in \mathscr{N}$ or $s^n \in \mathscr{N}$). Let $\mathfrak{R}$ be a Krasner hyperring, $\mathscr{N}$ be a proper hyperideal of $\mathfrak{R}$ and $\phi : Id(\mathfrak{R}) \to Id(\mathfrak{R}) \cup \{\emptyset\}$ be a function. $\mathscr{N}$ is called a ϕ-r-hyperideal (ϕ-pr-hyperideal) if $a \cdot b \in \mathscr{N} - \phi(\mathscr{N})$ with $ann(a) = 0$ implies that $b \in \mathscr{N}$ ($b^n \in \mathscr{N}$), for $a,b \in \mathfrak{R}$.

In this paper, we first study δ-r-hyperideals of commutative Krasner hyperrings. We present the main theorem on δ-r-hyperideals. We demonstrate that the intersection of δ-r-hyperideals is a δ-r-hyperideal. Therefore, we show that the image and the inverse image of a δ-r-hyperideal is also a δ-r-hyperideal. In the last section, we investigate some properties of ϕ-δ-r-hyperideals. We prove that if M_i is a directed set of ϕ-δ-r-hyperideals of $\mathfrak{R}$, then the union of M_i is a ϕ-δ-r hyperideal. In addition, we show that the image and the inverse image of a ϕ-δ-r-hyperideal is a ϕ-δ-r-hyperideal. Lastly, we explore the relation between von Neumann regular hyperideals [19] and pure hyperideals.

2. δ-r-HYPERIDEALS

In this paper, $(\mathfrak{R},+,\cdot)$ denotes a commutative Krasner hyperring with nonzero identity. $r(\mathfrak{R})$ means the set of all regular elements of $\mathfrak{R}$.

Remember that an expnsion of a hyperideal, or simply hyperideal expansion, is a function δ that assigns each hyperideal $\mathscr{N}$ of a hyperring $\mathfrak{R}$ to another hyperideal $\delta(\mathscr{N})$ of the same hyperring if the following statements hold:

i) $\mathscr{N} \subseteq \delta(\mathscr{N})$,

ii) $\mathscr{P} \subseteq Q$ implies $\delta(\mathscr{P}) \subseteq \delta(Q)$ for $\mathscr{P}, Q$ hyperideals of $\mathfrak{R}$ [27].

Moreover, let δ be a hyperideal exapansion. A hyperideal $\mathscr{N}$ of $\mathfrak{R}$ is said to be δ-primary if $a \cdot b \in \mathscr{N}$ and $a \notin \mathscr{N}$ imply $b \in \delta(\mathscr{N})$, for all $a,b \in \mathfrak{R}$ [27].

Definition 2.1. *Let δ be an expansion of hyperideals, $\mathfrak{R}$ be a Krasner hyperring, and $\mathscr{N}$ be a proper hyperideal of $\mathfrak{R}$. $\mathscr{N}$ is called a δ-r-hyperideal if $a \cdot b \in \mathscr{N}$ with $ann(a) = 0$ implies $b \in \delta(\mathscr{N})$, for all $a,b \in \mathfrak{R}$.*

Example 2.2. *(1) If $\mathscr{N}$ is a r-hyperideal with $\delta_0(\mathscr{N}) = \mathscr{N}$, then $\mathscr{N}$ is a δ_0-r-hyperideal.*
(2) If $\mathscr{N}$ is a pr-hyperideal with $\delta_1(\mathscr{N}) = \sqrt{\mathscr{N}}$, then $\mathscr{N}$ is a δ_1-r-hyperideal.
(3) If $\mathscr{N}$ is a r-hyperideal with $\delta_r(\mathscr{N}) = \mathfrak{R}$, then $\mathscr{N}$ is a δ_r-r-hyperideal.

Remark 2.3. *Let δ be an expansion of hyperideals. It is clear that the following statement hold:*
i) $\mathscr{N}$ is a ϕ-r-hyperideal $\Rightarrow$ $\mathscr{N}$ is a r-hyperideal $\Rightarrow$ $\mathscr{N}$ is a δ-r-hyperideal.
ii) If $\mathscr{N}$ is a δ-r-hyperideal, then $(\mathscr{N} : \mathscr{M})$ is a δ-r-hyperideal, for any subset $\mathscr{M}$ of $\mathfrak{R}$.
iii) If $\delta(\mathscr{N})$ is a δ-r-hyperideal, then $\mathscr{N}$ is a δ-r-hyperideal.

Remark 2.4. *i) Assume that δ and γ are hyperideal expansions and $\delta(\mathscr{N}) \subseteq \gamma(\mathscr{N})$, for each proper hyperideal $\mathscr{N}$. Then any δ-r-hyperideal is a γ-r-hyperideal.*
ii) Let δ_1 and δ_2 be two hyperideal expansions, $\mathscr{N}$ be a r-hyperideal and $\delta(\mathscr{N}) = \delta_1(\mathscr{N}) \cap \delta_2(\mathscr{N})$. Then δ is also a hyperideal expansion.
iii) Let δ be an expansion of hyperideals. $E_\delta(\mathscr{P}) = \bigcap\{\mathscr{M} \in Id(\mathfrak{R}) : \mathscr{P} \subseteq \mathscr{M}, \mathscr{M}$ is a δ-r-hyperideal$\}$. Then E_δ is a hyperideal expansion.

For all $\mathscr{P} \in Id(\mathfrak{R})$, by the definition of $E_\delta(\mathscr{P})$, it is clear that $\mathscr{P} \subseteq E_\delta(\mathscr{P})$. We show that for any $K,L \in Id(\mathfrak{R})$, if $K \subseteq L$, then $E_\delta(K) \subseteq E_\delta(L)$. Indeed: for any $K,L \in Id(\mathfrak{R})$, if $K \subseteq L$, then the δ-r-hyperideal that contains L also contains K. Furthermore, there may be δ-r-hyperideals that contain L but do not contain K. As a result, $E_\delta(K) \subseteq E_\delta(L)$.

Theorem 2.5. *Let $\mathfrak{R}$ satisfy the strong annihilator condition, $\mathcal{N}$ be a proper hyperideal, δ be an expansion of hyperideals. $\mathcal{P}$ is a δ-r-hyperideal if and only if for finitely generated hyperideal $\mathcal{N}$ and every hyperideal $\mathcal{M}$ of $\mathfrak{R}$ such that $\mathcal{N}\cdot\mathcal{M}\subseteq\mathcal{P}$ and $ann(\mathcal{N})=0$ implies $\mathcal{M}\subseteq\delta(\mathcal{P})$.*

Proof. Assume that $\mathcal{P}$ is a δ-r-hyperideal, $\mathcal{N}\cdot\mathcal{M}\subseteq\mathcal{P}$, $ann(\mathcal{N})=0$ and $\mathcal{M}\nsubseteq\delta(\mathcal{P})$. Then there exists $a\in\mathcal{N}$, such that $ann(a)=ann(\mathcal{N})=0$ and $b\in\mathcal{M}-\delta(\mathcal{P})$. Thus, $a\cdot b\in\mathcal{N}\cdot\mathcal{M}\subseteq\mathcal{P}$, $ann(a)=0$ and $b\notin\delta(\mathcal{P})$. This is a contradiction.

Conversely, let $a\cdot b\in\mathcal{P}$ and $ann(a)=0$. Then $<a>\cdot<b>\subseteq\mathcal{P}$ and $ann(<a>)=0$. Because of the assumption, $<b>\subseteq\delta(\mathcal{P})$. Thus, $b\in\delta(\mathcal{P})$. Hence, $\mathcal{P}$ is a δ-r-hyperideal. □

Theorem 2.6. *Let δ be a hyperideal expansion. Then the following statements are equivalent:*
(i) $\mathcal{N}$ is a δ-r-hyperideal;
(ii) $(\mathcal{N}:a)\subseteq\delta(\mathcal{N})$, for $a\in r(\mathfrak{R})$;
(iii) $(\mathcal{N}:\mathcal{P})\subseteq\delta(\mathcal{N})$, $\mathcal{P}$ is a hyperideal of $\mathfrak{R}$ such that $\mathcal{P}\cap r(\mathfrak{R})\nsubseteq\mathcal{N}$.

Proof. (i)⇒(ii) Let $x\in(\mathcal{N}:a)$, for $a\in r(\mathfrak{R})-\delta(\mathcal{N})$. Then $x\cdot a\in\mathcal{N}$ and $ann(a)=0$. Since $\mathcal{N}$ is δ-r-hyperideal, $x\in\delta(\mathcal{N})$.

(ii)⇒(iii) It is obvious.

(iii)⇒(i) Let $a\cdot x\in\mathcal{N}$ and $\mathcal{P}\cap r(\mathfrak{R})\nsubseteq\mathcal{N}$. Then for an element x of $\mathfrak{R}$, $x\in\mathcal{P}\cap r(\mathfrak{R})$ and $x\notin\mathcal{N}$. This means $ann(x)=0$ and $a\in(\mathcal{N}:\mathcal{P})\subseteq\delta(\mathcal{N})$. Thus, $\mathcal{N}$ is a δ-r-hyperideal. □

Theorem 2.7. *Let δ be a hyperideal expansion that preserves the intersection. If $Q_1,Q_2,...,Q_n$ are δ-r-hyperideals of $\mathfrak{R}$ and $\mathcal{P}=\delta(Q_i)$ for all i, then $Q=\bigcap_{i=1}^{n}Q_i$ is a δ-r-hyperideal.*

Proof. Let δ be a hyperideal expansion that preserves the intersection, $x\cdot y\in Q$ and $ann(x)=0$. Then $x\cdot y\in Q_k$ for some k. Since Q_k is a δ-r-hyperideal, then $y\in\delta(Q_k)$. $\delta(Q)=\delta(\bigcap_{i=1}^{n}Q_i)=\bigcap_{i=1}^{n}\delta(Q_i)=\mathcal{P}=\delta(Q_k)$. Hence, $y\in\delta(Q)$. □

Theorem 2.8. *Let δ be global, and $\varphi:\mathfrak{R}\to S$ be a good epimorphism and $\mathcal{N}$ be a δ-r-hyperideal of S. Then $\varphi^{-1}(\mathcal{N})$ is δ-r-hyperideal of $\mathfrak{R}$.*

Proof. Let $a\cdot b\in\varphi^{-1}(\mathcal{N})$ and $ann(a)=0$, for $a,b\in\mathfrak{R}$. Then $\varphi(a\cdot b)=\varphi(a)\cdot\varphi(b)\in\mathcal{N}$ and $ann(\varphi(a))=0$. Since $\mathcal{N}$ is a δ-r-hyperideal, then $\varphi(b)\in\delta(\mathcal{N})$. Thus, $b\in\varphi^{-1}(\delta(\mathcal{N}))=\delta(\varphi^{-1}(\mathcal{N}))$. Consequently, $\varphi^{-1}(\mathcal{N})$ is a δ-r-hyperideal of $\mathfrak{R}$. □

Theorem 2.9. *Let $\varphi:\mathfrak{R}\to S$ be a good epimorphism, and $\mathcal{N}$ be a hyperideal such that* $\ker(\varphi)\subseteq\mathcal{N}$. *Then, $\mathcal{N}$ is a δ-primary r-hyperideal of S if and only if $\varphi(\mathcal{N})$ is δ-primary r-hyperideal of $\mathfrak{R}$.*

Proof. If $\varphi(\mathcal{N})$ is a δ-r-hyperideal, since $\mathcal{N}=\varphi^{-1}(\varphi(\mathcal{N}))$, by the previous theorem, then $\mathcal{N}$ is a δ-r-hyperideal.

Let $a,b\in S$, $a\cdot b\in\varphi(\mathcal{N})$ and $ann(a)=0$. There are $x,y\in\mathfrak{R}$ such that $\varphi(x)=a$ and $\varphi(y)=b$. Since $a\cdot b=\varphi(x)\cdot\varphi(y)=\varphi(x\cdot y)\in\varphi(\mathcal{N})$, then $x\cdot y\in\varphi^{-1}(\varphi(\mathcal{N}))=\mathcal{N}$. We need to show that $ann(x)=0$. Let us suppose that $ann(x)\neq 0$. After that, there is $0\neq c\in\mathfrak{R}$ such that $x\cdot c=0$. Then $\varphi(x\cdot c)=\varphi(x)\cdot\varphi(c)=\varphi(0)=0$. Since $\varphi(c)\neq 0$, then $ann(\varphi(x))=ann(a)=0$. This is a contradiction. Thus $ann(x)=0$. Since $\mathcal{N}$ is a δ-r-hyperideal, then $y\in\delta(\mathcal{N})$. Hence, $b=\varphi(y)\in\varphi(\delta(\mathcal{N}))=\delta(\varphi(\mathcal{N}))$, since φ is onto. Therefore, $\varphi(\mathcal{N})$ is a δ-r-hyperideal. □

3. ϕ-δ-r-HYPERIDEALS

We define the reduction of hyperideals as a function ϕ that maps each hyperideal $\mathcal{N}$ of $\mathfrak{R}$ to another hyperideal $\phi(\mathcal{N})$ of the same hyperring $\mathfrak{R}$, satisfying the following axioms:

i) for all hyperideals $\mathcal{N}$ of $\mathfrak{R}$, $\phi(\mathcal{N})\subseteq\mathcal{N}$,

ii) if $\mathcal{N}\subseteq\mathcal{M}$, where $\mathcal{N}$ and $\mathcal{M}$ are hyperideals of $\mathfrak{R}$, then $\phi(\mathcal{N})\subseteq\phi(\mathcal{M})$.

Definition 3.1. *Let δ be a hyperideal expansion and ϕ a hyperideal reduction. A hyperideal $\mathscr{N}$ of $\mathfrak{R}$ is called a ϕ-δ-primary hyperideal if $a \cdot b \in \mathscr{N} - \phi(\mathscr{N})$, then $a \in \mathscr{N}$ or $b \in \delta(\mathscr{N})$, for all $a, b \in \mathfrak{R}$.*

Definition 3.2. *Let δ be a hyperideal expansion, ϕ be a hyperideal reduction. A proper hyperideal $\mathscr{N}$ of $\mathfrak{R}$ is called a ϕ-δ-r-hyperideal if $a \cdot b \in \mathscr{N} - \phi(\mathscr{N})$ with $ann(a) = 0$ implies $b \in \delta(\mathscr{N})$, for all $a, b \in \mathfrak{R}$.*

Let ϕ be a hyperideal reduction and δ be a hyperideal expansion. Then:
$\mathscr{N}$ is a ϕ-δ_0-r-hyperideal $\Leftrightarrow$ $\mathscr{N}$ is a ϕ-r-hyperideal,
$\mathscr{N}$ is a ϕ-δ_1-r-hyperideal $\Leftrightarrow$ $\mathscr{N}$ is a ϕ-pr-hyperideal,
$\mathscr{N}$ is a δ-r-hyperideal $\Leftrightarrow$ $\mathscr{N}$ is a $\phi_\emptyset$-δ-r hyperideal,
$\mathscr{N}$ is a r-hyperideal $\Leftrightarrow$ $\mathscr{N}$ is a $\phi_\emptyset$-δ_0-r-hyperideal,
$\mathscr{N}$ is a pr-hyperideal $\Leftrightarrow$ $\mathscr{N}$ is a $\phi_\emptyset$-δ_1-r-hyperideal.

Remark 3.3. *i) If δ, γ are two hyperideal expansions with $\delta \leq \gamma$ and ϕ is any hyperideal reduction, then every ϕ-δ-r-hyperideal of $\mathfrak{R}$ is a ϕ-γ-r-hyperideal.*

ii) If $\delta_1, \ldots, \delta_n$ are hyperideal expansions, then $\delta = \bigcap_{i=1}^{n} \delta_i$ is also a hyperideal expansion.

Proposition 3.4. *Let $\{M_i : i \in D\}$ be a directed set of ϕ-δ-r-hyperideals of $\mathfrak{R}$, where ϕ is a hyperideal reduction and δ is a hyperideal expansion. Then the hyperideal $\mathscr{M} = \bigcup_{i \in D} M_i$ is a ϕ-δ-r-hyperideal.*

Proof. Let $a \cdot b \in \mathscr{M} - \phi(\mathscr{M})$ with $ann(a) = 0$, for $a, b \in \mathfrak{R}$. Since $a \cdot b \notin \phi(\mathscr{M})$, then $a \cdot b \notin \phi(\mathscr{M})$. Let $i \in D$ be such that $a \cdot b \in M_i - \phi(M_i)$. Since $ann(a) = 0$ and M_i is a ϕ-δ-r-hyperideal, then $b \in \delta(M_i) \subseteq \delta(\mathscr{M})$. Hence, $\mathscr{M}$ is a ϕ-δ-r-hyperideal. □

Theorem 3.5. *Let $\mathscr{N}$ be a proper hyperideal of $\mathfrak{R}$. Suppose that ϕ is a hyperideal reduction and δ is a hyperideal expansion. Then the following statements are equivalent:*
(i) $\mathscr{N}$ is a ϕ-δ-r-hyperideal;
(ii) $(\mathscr{N} : a) \subseteq \delta(\mathscr{N}) \cup (\phi(\mathscr{N}) : a)$, for $a \in r(\mathfrak{R})$;
(iii) $(\mathscr{N} : \mathscr{P}) \subseteq \delta(\mathscr{N}) \cup (\phi(\mathscr{N}) : \mathscr{P})$, $\mathscr{P}$ is a hyperideal of $\mathfrak{R}$ such that $\mathscr{P} \cap r(\mathfrak{R}) \nsubseteq \mathscr{N}$.

Proof. (i)⇒(ii) Let $x \in (\mathscr{N} : a)$. Then $x \cdot a \in \mathscr{N}$. If $x \cdot a \in \phi(\mathscr{N})$, then $x \in (\phi(\mathscr{N}) : a)$. If $x \cdot a \notin \phi(\mathscr{N})$, then $x \in \delta(\mathscr{N})$.

(ii)⇒(iii) It is obvious.

(iii)⇒(i) Let $a \cdot x \in \mathscr{N} - \phi(\mathscr{N})$ and $\mathscr{P} \cap r(\mathfrak{R}) \nsubseteq \mathscr{N}$. Then for an element x of $\mathfrak{R}$, we have $x \in \mathscr{P} \cap r(\mathfrak{R})$ and $x \notin \mathscr{N}$. This means $ann(x) = 0$ and $a \in (\mathscr{N} : \mathscr{P}) - (\phi(\mathscr{N}) : \mathscr{P})$. Since $a \notin (\phi(\mathscr{N}) : \mathscr{P})$, then $a \in \delta(\mathscr{N})$. □

Theorem 3.6. *Let $\mathfrak{R}$ satisfy the strong annihilator condition and $\mathscr{N}$ be a proper hyperideal. $\mathscr{P}$ is a ϕ-δ-r-hyperideal if and only if $\mathscr{N} \cdot \mathscr{M} \subseteq \mathscr{P} - \phi(\mathscr{N})$ and $ann(\mathscr{N}) = 0$ implies $\mathscr{M} \subseteq \delta(\mathscr{P})$, where $\mathscr{N}$ is a finitely generated hyperideal and $\mathscr{M}$ is a hyperideal of $\mathfrak{R}$.*

Proof. Assume that $\mathscr{P}$ is a ϕ-δ-r-hyperideal, $\mathscr{N} \cdot \mathscr{M} \subseteq \mathscr{P} - \phi(\mathscr{N})$, $ann(\mathscr{N}) = 0$ and $\mathscr{M} \nsubseteq \delta(\mathscr{P})$. Then there exists $a \in \mathscr{N}$, such that $ann(a) = ann(\mathscr{N}) = 0$ and $b \in \mathscr{M} - \delta(\mathscr{P})$. Thus, $a \cdot b \in \mathscr{N} \cdot \mathscr{M} \subseteq \mathscr{P} - \phi(\mathscr{N})$, $ann(a) = 0$ and $b \notin \delta(\mathscr{P})$. This is a contradiction.

Conversely, let $a \cdot b \in \mathscr{P} - \phi(\mathscr{N})$ and $ann(a) = 0$. Then $< a > \cdot < b > \subseteq \mathscr{P} - \phi(\mathscr{N})$ and $ann(< a >) = 0$. Because of the assumption, $< b > \subseteq \delta(\mathscr{P})$. Thus, $b \in \delta(\mathscr{P})$. Hence, $\mathscr{P}$ is a ϕ-δ-r-hyperideal. □

Theorem 3.7. *Let ϕ be a hyperideal reduction and δ be a hyperideal expansion. If $\mathscr{P}$ is a ϕ-δ-r-hyperideal of $\mathfrak{R}$ and $\phi(\mathscr{P} : a) = (\phi(\mathscr{P}) : a)$, for $a \in \mathfrak{R}$, then $(\mathscr{P} : a)$ is also a ϕ-δ-r-hyperideal of $\mathfrak{R}$.*

Proof. Let $a \cdot b \in (\mathscr{P} : a) - \phi(\mathscr{P} : a)$ with $ann(x) = 0$. Then $x \cdot y \cdot a \in \mathscr{P} - \phi(\mathscr{P})$ and $ann(x \cdot a) = 0$. Since $\mathscr{P}$ is a ϕ-δ-r-hyperideal and $\phi(\mathscr{P} : a) = (\phi(\mathscr{P}) : a)$, then $b \in \delta(\mathscr{P}) \subseteq \delta(\mathscr{P} : a)$. Hence, $(\mathscr{P} : a)$ is a ϕ-δ-r-hyperideal of $\mathfrak{R}$. □

A reduction is said to be global if for any hyperring homomorphism $\varphi : \mathfrak{R} \to S$, $\phi(\varphi^{-1}(\mathscr{N})) = \varphi^{-1}(\phi(\mathscr{N}))$, for all $\mathscr{N} \in Id(S)$. For example, the reductions ϕ_1 and ϕ_2 are both global.

Theorem 3.8. *Let ϕ be a hyperideal reduction and δ be a hyperideal expansion, both of which are global. If $\varphi : \mathfrak{R} \to S$ is a good epimorphism, then for any ϕ-δ-r-hyperideal $\mathscr{N}$ of S, $\varphi^{-1}(\mathscr{N})$ is a ϕ-δ-r-hyperideal of $\mathfrak{R}$.*

Proof. Suppose that $\mathscr{N}$ is a ϕ-δ-r-hyperideal of S. Let $a \cdot b \in \varphi^{-1}(\mathscr{N}) - \phi(\varphi^{-1}(\mathscr{N}))$ with $ann(a) = 0$. Then $\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b) \in \mathscr{N} - \varphi(\mathscr{N})$. If $ann(\varphi(a)) \neq 0$, there is a $0 \neq \varphi(x) \in S$, for $x \in \mathfrak{R}$, such that $\varphi(a) \cdot \varphi(x) = \varphi(a \cdot x) = 0$. This means there is a $0 \neq x \in \varphi^{-1}(\mathscr{N})$ such that $a \cdot x = 0$, that is a contradiction. Hence, $ann(\varphi(a)) = 0$. Since $\mathscr{N}$ is a ϕ-δ-r-hyperideal, then $\varphi(b) \in \delta(\mathscr{N})$. And so, $b \in \varphi^{-1}(\delta(\mathscr{N}))$. □

Theorem 3.9. *Let ϕ be a hyperideal reduction and δ be a hyperideal expansion such that both are global. Let $\varphi : \mathfrak{R} \to S$ be a good epimorphism. Then any ϕ-δ-r-hyperideal $\mathscr{M}$ of $\mathfrak{R}$ contains* $\ker\varphi$ *if and only if $\varphi(\mathscr{M})$ is a ϕ-δ-r-hyperideal of S.*

Proof. If $\varphi(\mathscr{M})$ is a ϕ-δ-r-hyperideal of S, then, by previous theorem, $\mathscr{M} = \varphi^{-1}(\varphi(\mathscr{M}))$ is a ϕ-δ-r-hyperideal of $\mathfrak{R}$. Conversely, let $b_1 \cdot b_2 \in \varphi(\mathscr{M}) - \phi(\varphi(\mathscr{M}))$ and $ann(b_1) = 0$ for $b_1, b_2 \in S$. Since φ is onto, then there exist $a_1, a_2 \in \mathfrak{R}$ such that $b_1 = \varphi(a_1)$ and $b_2 = \varphi(a_2)$. Then $b_1 \cdot b_2 = \varphi(a_1) \cdot \varphi(a_2) = \varphi(a_1 \cdot a_2) = \varphi(x) \in \varphi(\mathscr{M}) - \phi(\varphi(\mathscr{M}))$, for some $x \in \mathscr{M}$. $0 \in \varphi(a_1 \cdot a_2) - \varphi(x) = \varphi(a_1 \cdot a_2 - x)$. Then, there is a $t \in a_1 \cdot a_2 - x$ such that $\varphi(t) = 0$: We have $a_1 \cdot a_2 \in t + x \subseteq \ker\varphi + \mathscr{M} \subseteq \mathscr{M} + \mathscr{M} \subseteq \mathscr{M}$: Thus, $a_1 \cdot a_2 \in \mathscr{M} - \phi(\mathscr{M})$. Let $ann(a_1) \neq 0$: Then there exists $c(\neq 0) \in \mathfrak{R}$ such that $a_1 \cdot c = 0$. So, $\varphi(a_1 \cdot c) = \varphi(a_1) \cdot \varphi(c) = \varphi(0) = 0$. Since $\varphi(c) \neq 0$, then $ann(\varphi(a_1)) = ann(b_1) = 0$. This is a contradiction. So, this implies that $ann(a_1) = 0$: Since $\mathscr{M}$ is a ϕ-δ-r-hyperideal, then $a_2 \in \delta(\mathscr{M})$. We have $b_2 = \varphi(a_2) \in \varphi(\delta(\mathscr{M}))$. Since φ is onto, then $\delta(\mathscr{M}) = \delta(\varphi^{-1}(\varphi(\mathscr{M}))) = \varphi^{-1}(\delta(\varphi(\mathscr{M})))$, therefore $\varphi(\delta(\mathscr{M})) = \delta(\varphi(\mathscr{M}))$. Therefore, $b_2 \in \delta(\varphi(\mathscr{M}))$. □

Remark 3.10. *By Theorem 3.8, for a global hyperideal expansion δ, if $\varphi : \mathfrak{R} \to S$ is a good epimorphism and* $\ker\varphi \subseteq \mathscr{N}$, *for some hyperideal $\mathscr{N}$ of $\mathfrak{R}$, then $\varphi(\delta(\mathscr{M})) = \delta(\varphi(\mathscr{M}))$. Therefore, if φ is an isomorphism, then $\varphi(\delta(\mathscr{M})) = \delta(\varphi(\mathscr{M}))$ for all $\mathscr{N} \in Id(\mathfrak{R})$.*

Corollary 3.11. *Let ϕ be a hyperideal reduction and δ be a hyperideal expansion, all of which are global. Let $\mathscr{N}$, $\mathscr{M}$ be any hyperideals of $\mathfrak{R}$ such that $\mathscr{M}$ contains $\mathscr{N}$. Then $\mathscr{M}/\mathscr{N}$ is a ϕ-δ-r-hyperideal of $\mathfrak{R}/\mathscr{N}$ if and only if $\mathscr{M}$ is a ϕ-δ-r-hyperideal of $\mathfrak{R}$.*

Proof. Let $\varphi : \mathfrak{R} \to \mathfrak{R}/\mathscr{N}$ be a good epimorphism with $\ker\varphi = \mathscr{N} \subseteq \mathscr{M}$. By Theorem 10, $\mathscr{M}/\mathscr{N}$ is a ϕ-δ-r-hyperideal of $\mathfrak{R}/\mathscr{N}$ if and only if $\mathscr{M}$ is a ϕ-δ-r-hyperideal of $\mathfrak{R}$. □

Definition 3.12. *Let ϕ be a hyperideal reduction and δ be a hyperideal expansion. Let $\mathscr{N}$ be a proper hyperideal.*

(1) If for every $a \in \mathscr{N} - \phi(\mathscr{N})$, there is $b \in \delta(\mathscr{N})$ such that $a = a \cdot b$, then $\mathscr{N}$ is called a ϕ-δ-pure hyperideal.

(2) If for every $a \in \mathscr{N} - \phi(\mathscr{N})$, there is $b \in \delta(\mathscr{N})$ such that $a = a^2 \cdot b$, then $\mathscr{N}$ is called a ϕ-δ-von Neumann regular hyperideal.

Corollary 3.13. *i) Every ϕ-δ-pure hyperideal is a ϕ-δ-r hyperideal.*

ii) Every ϕ-δ-von Neumann regular hyperideal is a ϕ-δ-r-hyperideal.

Proof. Using Definition 3.12, we can simply prove that any ϕ-δ-pure hyperideal is a ϕ-δ-r-hyperideal. Moreover, each ϕ-δ-von Neumann regular hyperideal is likewise a ϕ-δ-r-hyperideal. □

4. CONCLUSIONS

In this study, we defined the expansion of r-hyperideals and extended this concept to ϕ-δ-r-hyperideals over the commutative Krasner hyperrings with nonzero identity. We investigated some of their essential characteristics and provided several examples. Furthermore, we examined the relations between these structures. The overall framework of these structures was explained, providing a number of major conclusions. We obtained some specific results explaining the structures. For example, we indicated that when $\mathfrak{R}$ satisfies the strong annihilator condition, $\mathscr{N}$ is a proper hyperideal and δ is an expansion of hyperideals, $\mathscr{P}$ is a δ-r-hyperideal if and only if $\mathscr{N} \cdot \mathscr{M} \subseteq \mathscr{P}$ and $ann(\mathscr{N}) = 0$ implies $\mathscr{M} \subseteq \delta(\mathscr{P})$ for finitely generated hyperideal $\mathscr{N}$ and hyperideal $\mathscr{M}$ of $\mathfrak{R}$. Then we demonstrated that a similar result is satisfied for ϕ-δ-r-hyperideals of $\mathfrak{R}$.

This study offers a significant advance in the classification of hyperideals in Krasner hyperrings. Following this paper, it is feasible to investigate additional algebraic hyperstructures, such as Krasner (m,n)-hyperrings, and do research on their properties. Based on our study, we suggest some open problems for future work, such as describing 2-absorbing δ-r-hyperideals and considering δ-r-subhypermodules and ϕ-δ-r-subhypermodules.

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REFERENCES

1. Asokkumar, A. and Velrajan, M. Von Neumann regularity on Krasner hyperring. In *Algebra, Graph Theory and Their Applications* (Tamizh Chelvam, T., Somasundaram, S. and Kala, R., eds). Narosa Publishing House, New Delhi, 2010.
2. Al Tahan, M. and Davvaz, B. On the existence of hyperrings associated to arithmetic functions. *J. Number Theory*, 2017, **174**, 136–149.
3. Al Tahan, M. and Davvaz, B. Strongly regular relations of arithmetic functions. *J. Number Theory*, 2018, **187**, 391–402.
4. Anderson, D. D. and Bataineh, M. Generalizations of prime ideals. *Commun. Algebra*, 2008, **36**(2), 686–696.
5. Corsini, P. *Prolegomena of Hypergroup Theory*. Aviani, Tricesimo, 1993.
6. Corsini, P. and Leoreanu, V. *Applications of Hyperstructure Theory*. (ADMA 5). Kluwer Academic Publishers, Dordrecht, 2003.
7. Darani, A. Y. Generalizations of primary ideals in commutative rings. *Novi Sad J. Math.*, 2012, **42**(1), 27–35.
8. Davvaz, B. *Polygroup Theory and Related Systems*. World Scientific, Hackensack, 2013.
9. Davvaz, B. and Leoreanu-Fotea, V. *Hyperring Theory and Applications*. International Academic Press, Palm Harbor, 2007.
10. Davvaz, B. and Vougiouklis, T. *A Walk Through Weak Hyperstructures: Hv-structures*. WSPC, Hackensack, 2019.
11. Davvaz, B. and Leoreanu-Fotea, V. *Hypergroup Theory*. WSPC, Hackensack, 2022.
12. Davvaz, B. and Salasi, A. A realization of hyperrings. *Commun. Algebra*, 2006, **34**(12), 4389–4400.
13. Zhao, D. δ-primary ideals of commutative rings. *Kyungpook Math. J.*, 2001, **41**(1), 17–22.
14. Guan, H., Kaya, E., Bolat, M., Onar, S., Ersoy, B. A. and Hila, K. ϕ-δ-primary hyperideals in Krasner hyperrings. *Math. Probl. Eng.*, 2022, **2022**, 1192684.
15. Huckaba, J. A. *Commutative Rings With Zero Divisors*. Marcel Dekker, Inc., New York, 1988.
16. Jaber, A. Properties of ϕ-δ-primary and 2-absorbing δ-primary ideals of commutative rings. *Asian-European J. Math.*, 2020, **13**(1), 2050026.
17. Krasner, M. A class of hyperrings and hyperfields. *Int. J. Math. Math. Sci.*, 1983, **6**(2), 307–312.
18. Krasner, M. Approximation des corps valués complets de charactéristique $p \neq 0$ par ceux de caractéristique 0. In *Colloque d'algebre superieure*, Bruxelles 19–22 December 1956. Center Belge de Recherches Mathematiques, 1957, 129–206.
19. Marty, F. Sur une generalization de la notion de groupe. In *Proceedings of the 8th Scandinavian Congress of Mathematicians, Stockholm, Sweden, 14–18 August, 1934*. Ohlssons, 45–49.
20. Massouros, C. and Massouros, G. An overview of the foundations of the hypergroup theory. *Mathematics*, 2021, **9**(9), 1014.
21. Massouros, C. On the theory of hyperrings and hyperfields. *Algebra Log.*, 1985, **24**, 728–742.

22. Mittas, J. Hypergroupes canoniques. *Math. Balkanica*, 1972, **2**, 165–179.
23. Massouros, G. and Massouros, C. Hypercompositional algebra, computer science and geometry. *Mathematics*, 2020, **8**(8), 1338.
24. Mohamadian, R. r-ideals in commutative rings, *Turkish J. Math*., 2015, **39**(5), 733–749.
25. Nakassis, A. Recent results in hyperring and hyperfield theory. *Int. J. Math. Math. Sci*., 1988, **11**(2), 209–220.
26. Omidi, S., Davvaz, B. and Zhan, J. Some properties of n-hyperideals in commutative hyperrings, *J. Algebraic Hyperstruct. Logical Algebras*, 2020, **1**(2), 23–30.
27. Ozel Ay, E. δ-primary hyperideals on commutative hyperrings, *Int. J. Math. Math. Sci*., 2017, **2017**, 5428160.
28. Stefanescu, M. 2006. Constructions of hyperfields and hyperrings. *Stud. Cercet. Stiint. Ser. Mat. Univ. Bacau*, 16, 563–571.
29. Ugurlu, E. A. Generalizations of r-ideals of commutative rings. *J. Interdisciplinary Math*., 2021, **24**(8), 2283–2293.
30. Vougiouklis, T. *Hyperstructures and Their Representations*. Hadronic Press, Palm Harbor, 1994.
31. Vougiouklis, T. The fundamental relation in hyperrings. The general hyperfield. In *Proceedings of the 4th International Congress on Algebraic Hyperstructures and Applications (AHA), 1990*. World Scientific, 1991, 203–211.
32. Xu, P., Bolat, M., Kaya, E., Onar, S., Ersoy, B. A. and Hila, K. r-hyperideals and generalizations of r-hyperideals in Krasner hyperrings. *Math. Probl. Eng*., 2022, **2022**, 7862425.
33. Xu, P., Bolat, M., Kaya, E., Onar, S., Ersoy, B. A. and Hila, K. δ-r-hyperideals and ϕ-δ-r-hyperideals of commutative Krasner hyperrings. 2021. https://arxiv.org/abs/2112.05513