Proceedings of the Estonian Academy of Sciences,
2023, **72**, 1, 77–95
https://doi.org/10.3176/proc.2023.1.08
Available online at www.eap.ee/proceedings

MATHEMATICAL PHYSICS

Generalized approach to Galileo's swiftest descent problem on a circle

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Received 2 July 2022, accepted 11 August 2022, available online 10 February 2023

Abstract. Nearly 400 years ago Galileo Galilei posed a conjecture about the fastest descent on the lower half of a circle. Namely, he assumed that the descent along the circular arc itself is faster than along any broken line of chords. Galileo studied the case when the initial speed is zero, and the paths end in the lowest point of the circle, but he made a guess that the final conclusion remains the same if the particle initially at rest falls to a starting point where the speed is not zero. Galileo was right, as is now rigorously proved. However, this intuitive idea cannot be extended to a starting point on the upper half of a circle. Indeed, as we demonstrate, the fastest descent would then correspond to a finite (not infinite!) number of connected chords. Moreover, we present an extrapolation method which can be applied to determine the optimal number and the positions of all these chords for any starting point on a circle.

Keywords: path optimization problems, multidimensional minimization, kinematics of a particle, fastest descent on a circle, brachistochrone problem.

1. INTRODUCTION

Galileo's scientific testament *Discourses and Mathematical Demonstrations Concerning Two New Sciences* (referred to as *Discorsi* below) includes a famous theorem (Proposition 6, Theorem 6; known as the law of chords: "If, from the highest or lowest point of a vertical circle, any inclined planes whatever are drawn to its circumference, the times of descent through these will be equal." [1, p. 178]. Galileo's proof of this theorem was most ingenious: he just combined the laws of uniformly accelerating motion with some elementary geometrical considerations (see [2], p. 138, for more details). Using the law of chords, Galileo obtained an even more important theoretical result. Namely, he proved that any two-chord path on a lower quadrant of a circle is faster than the straight one-chord descent from the same starting point to the same end point, if the particle is initially at rest [1, Proposition 36, Theorem 22, p. 211]. Galileo's proof of Proposition 36 was correct but rather complicated. A more compact and more general proof is given in Appendix A of this article (see [3]).

The essence of Galileo's result is most clearly expressed in the Scholium to Proposition 36: "From the things demonstrated, it appears that one can deduce that the swiftest movement of all from one terminus to the other is not through the shortest line of all, which is the straight line . . . , but through the circular arc." [1, pp. 212–213]. This was an astonishing insight, and again, Galileo was right. However, quite understandably, he did not give a proof of this conjecture, as the problem is not at all trivial. Indeed, as Galileo obviously was aware of, and as has been pointed out by several authors [4,5], Proposition 36 cannot be extended to a situation where the particle is not initially at rest. A rigorous analytical proof for this case has been given only very recently [6].

The above analysis only concerns the descent of a particle in the lower quadrant of a circle. The goal of this article is to go beyond that limit and examine the descent along any n-chord path on the whole vertical semicircle. In Section 2, we prove that the fastest path always consists of a finite number ($1 \leq n < \infty$) of connected chords, if $\varphi_n > 90°$. This is the main theoretical result which is specified in the subsequent sections. It will be shown that for any $n > 0$ it is possible to determine, step-by-step, the upper and the lower bounds of the region where the n-chord path is the fastest. For example, a two-chord path ($n = 2$) is the fastest if $\varphi_2^{(0)} < \varphi_n < 120°$, the characteristic critical angle being $\varphi_2^{(0)} \approx 96.4672°$ (the subscript indicates the number of chords, while the superscript $^{(0)}$ indicates that $\varphi_2^{(0)}$ is the starting angle of that two-chord path). Analogously, a three-chord path ($n = 3$) is the fastest if $\varphi_3^{(0)} < \varphi_n < \varphi_2^{(0)}$, where $\varphi_3^{(0)} \approx 92.6179°$. The number of components of the fastest path will rapidly increase as φ_I approaches 90°, but it always remains finite if $\varphi_I > 90°$.

It is worth mentioning that the word "chord" was actually not used in the formulation of Proposition 36. In fact, this was an assertion about the descent along inclined planes on a cylindrical surface. Formally, as circle is the cross-section of a cylinder, and chords are the projections of corresponding inclined planes, the results of the analysis remain the same, but there is an important nuance: a freely falling particle cannot stay on a plane with the inclination angle above 90°. On the other hand, as will be shown in Section 2, the fastest path cannot include a chord with the inclination angle above 90°, if the particle starts to fall from any point in the upper quadrant of a circle. Thus, although we will study a particle's descent along a set of connected inclined planes on a cylindrical surface, it is justified to refer to the projections of these planes, i.e., to a corresponding set of connected chords on a circle. The end points of n chords are fixed by the angles $\varphi_F \equiv \varphi_0 < \varphi_1 < ... < \varphi_{n-1} < \varphi_n \equiv \varphi_I$, and we assume that a particle starts at rest at an angle φ_n, then drops to φ_{n-1}, φ_{n-2}, etc. In most cases we set the final angle to zero ($\varphi_0 = 0$), but this constraint is practical rather than critical, in order to express the results in a more compact form. However, from a physicist's point of view there is a much more important constraint:

$$\varphi_{n-1} \leq \pi - \varphi_n, \tag{1}$$

which excludes the inclination angles above 90°.

Comment. Constraint (1) is relevant to a freely falling particle on a cylindrical surface, but it can be ignored if we slightly modify the system to be studied. Suppose a bead of negligible mass is freely sliding on a wire of negligible thickness. Let that wire consist of n connected chords of a circle ($1 \leq n < \infty$). Then there is no need for the constraint (1), i.e., the inclination angles may exceed 90°. Nevertheless, as we demonstrate, the solution of the problem remains the same. Thus, the mathematical problem to be solved is as follows.

Statement of the problem: to devise a method that would enable, for any given starting angle $\varphi \in (90°, 180°)$ and a fixed final angle φ_0, to find the optimal number (n) of connected chords on a circle, and to determine all intermediate angles φ_i ($i = 1, 2, ..., n-1$), so that $\varphi_n = \varphi$ and the descent time $t(\varphi_n)$ is minimal.

Let us briefly explain what we expect to achieve. Suppose a bead is at rest at some initial point I on a circle, and let it slide (without friction, rotation and air drag) towards some final point F on the same circle. Suppose the path between I and F can be divided into one or more segments, so that each of them can either be a straight chord or an arc along the circle. All node points of the path (including I and F) can be located by corresponding angles, as shown in Fig. 1, and we assume that $0° \leq \varphi_F < \varphi_I \leq 180°$. For any fixed pair of the parameters φ_F and φ_I, there are infinitely many different paths, but our goal is to give the correct answers to the following questions. Out of all these possible paths, which will give the shortest descent time from point I to point F? In particular, how many intermediate points should there be and at what angles should they be located? The correct answer for the case $\varphi_I \leq 90°$ was already given (although not proven) by Galileo himself: the circular arc itself is the fastest path. In other words, if $\varphi_I \leq 90°$, then the optimal path consists of an infinite number of connected chords ($n \to \infty$). On the other hand, if $\varphi_I \geq 120°$, then the fastest path corresponds to a straight chord from point I to point F ($n = 1$) [7]. Thus, for these two regions

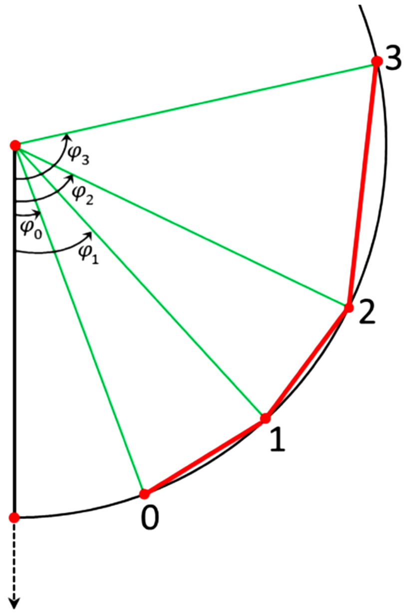

Fig. 1. An explanation to the statement of the problem and the notations used in this paper. A three-chord path from the initial point 3 to the final point 0 is characterized by the angles φ_i (i = 0;1;2;3) as shown in the figure. Note that the angles are measured from the downward vertical axis.

($\varphi_I \leq 90°$ and $\varphi_I \in [120°, 180°]$) the problem has already been solved, but the detailed theoretical analysis for the region $\varphi_I \in (90°, 120°)$ was missing, and this was the motivation for the present study.

In Section 3, a comparison is made between a 1-chord vs 2-chord and a 2-chord vs 3-chord descent, and we obtain some additional theoretical results. For example, a general formula is derived for the descent time along an arbitrary n-chord broken line ($\varphi_n > \varphi_{n-1} > ... > \varphi_0$). This is the basis for the extrapolation method described in Section 4. In principle, the approach can be applied to determine the optimal number and the exact positions of all components of the fastest multi-chord path for any starting point on a circle. In Section 5, we briefly discuss the possible practical value of the obtained results, and make a comparison with the true minimum-descent-time curve (a brachistochrone). Finally, a short summary is given in Section 6. The main statements will be rigorously proved, but some lengthy derivations will be given in the appendices of the article.

2. CAN GALILEO'S CLAIM BE EXTENDED?

Galileo's conjecture in the Scholium to Proposition 36 in his *Discorsi* was correct: if $0 < \varphi_I \leq 90°$, then the swiftest descent is achieved along the circular arc itself, i.e., via the infinite number of infinitesimal chords. One might think that this result could somehow be extended to the region $\varphi_I > 90°$, but this is not the case, as we are now going to demonstrate, step-by-step. First, in order to correctly pose the problem, we have to set some constraints.

As mentioned in the introduction, we expect that $\varphi_I \in (90°, 120°)$, i.e., if measured in radians,

$$\varphi_I = \pi/2 + \alpha, \alpha \in (0, \pi/6). \tag{2}$$

In this connection, we also set a constraint to the final angle φ_F. Namely, if φ_I is fixed, then

$$\varphi_F \in [0, \varphi_I - 4\alpha). \tag{3}$$

Combined with (2), this ensures that φ_F is non-negative for $\varphi_I \in (90°, 120°)$, but there is an even more important point. Namely, if the particle's trajectory is subjected to constraints (2) and (3), then $\varphi = \varphi_I - 4\alpha$

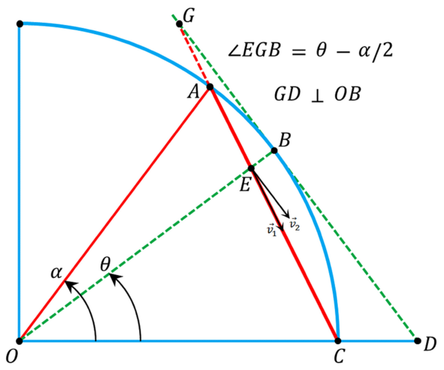

Fig. 2. The descent along the straight chord from point A to point C is faster than the descent along the circular arc connecting these points.

corresponds to the fastest chord-arc path between the points I and F (see the comment at the end of Section 2.1).

Remark. *Here, "chord-arc path" means that a particle at rest moves along a chord from the initial angle φ_I to an angle $\varphi \leq \pi/2$, and then along the arc from φ to the final angle φ_F.*

Obviously, with these constraints taken into account, the circular arc from φ_I to φ_F ($n \to \infty$) cannot be the fastest n-chord trajectory. Indeed, suppose a particle is initially at rest at an arbitrary point A in the upper quadrant of a circle. Then the descent along the straight chord from point A to the lowest point C of that quadrant (where $\theta = 0$) is faster than the descent along the circular arc AC (see Fig. 2). Let us prove this claim.

There is a one-to-one correspondence between an arbitrary point $B(\theta)$ on the arc AC and its projection $E(\theta)$ onto the chord AC. Consequently, we can explicitly compare the descent times along the chord AEC (trajectory 1) and along the arc ABC (trajectory 2). Suppose v_1 is the speed of a particle at point $E(\theta)$ on trajectory 1, and v_2 is the speed of a particle at point $B(\theta)$ on trajectory 2. Thus, applying the law of conservation of mechanical energy, and using some elementary geometry, we get

$$v_1^2 - v_2^2 = 4gL \frac{\sin\theta \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\alpha-\theta}{2}\right)}{\cos\left(\theta - \frac{\alpha}{2}\right)},$$

where $L = OA$ is the radius of the circle and g is the gravity acceleration constant.

The relationship between the descent times is determined by v_1 and $v_2 \cos(\theta - \alpha/2)$, and $v_1 > v_2 \cos(\theta - \alpha/2)$, if $\theta \neq \alpha$, while $v_1 = v_2 = 0$, if $\theta = \alpha$. Moreover, the differential arc length $ds = Ld\theta$ at any point on trajectory 2 is larger than its projection dl onto the chord AC. Therefore, the differential descent time at any point on trajectory 1 is shorter than the corresponding descent time for trajectory 2: $dt_1 \equiv dl/v_1 < dt_2 \equiv ds/[v_2 \cos(\theta - \alpha/2)]$. From this, we conclude that the descent from φ_I to φ_F becomes faster if the arc AC is replaced with the chord AC. This means that the whole circular arc cannot be the fastest trajectory. However, this does not automatically exclude the possibility that the fastest path may still consist of an infinite number of chords. Indeed, point C is not the end point of the particle's trajectory: there remains an arc from $\pi/2$ to φ_F. Therefore, we have to prove that whatever chord-arc path from φ_I to φ_F is not the fastest trajectory. To this end, we need some basic formulas for the descent time. They are given in the next subsection.

2.1. A few preliminary analytical results

Let 0, 1, 2 and 3 be the points on a circle, such that $0 \leq \varphi_0 < \varphi_1 < \varphi_2 \leq \varphi_3 \leq \pi$. Suppose a mass point at rest starts to fall from point 3, moving along an arbitrary path to point 2. The descent time along the chord from point 2 to point 1 can then be calculated as follows [6] (see also [3], Appendix B):

$$t_{21}^{(3)} = \frac{\sqrt{2}\sin\left(\frac{\varphi_2-\varphi_1}{2}\right)\cdot T_0}{\sqrt{\cos(\varphi_1)-\cos(\varphi_3)}+\sqrt{\cos(\varphi_2)-\cos(\varphi_3)}} = \frac{\sin(\alpha_2-\alpha_1)\cdot T_0}{\sqrt{F_1}+\sqrt{F_2}},\ \alpha_i \equiv \varphi_i/2\ (i=1,2,3), \tag{4}$$

where

$$T_0 \equiv 2\sqrt{L/g},\ F_i \equiv \sin^2(\alpha_3)-\sin^2(\alpha_i) = \sin(\alpha_3+\alpha_i)\sin(\alpha_3-\alpha_i),$$

L is the radius of the circle, and g is the gravity acceleration constant. Throughout this paper, T_0 is taken to be the unit of time, i.e., $T_0 = 1$.

The overall descent time for a chord-arc path from point 3 to point 0 therefore reads as

$$T(\alpha_2) = t_{32}^{(3)} + t_{20}^{(arc)} = \frac{\sin(\alpha_3-\alpha_2)}{\sqrt{F_2}} + \frac{1}{2}\int_{\phi_0}^{\phi_2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}. \tag{5}$$

Here, the second term

$$t_{20}^{(arc)} \equiv \frac{1}{2}\int_{\phi_0}^{\phi_2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} \tag{6}$$

corresponds to the arc from point 2 to point 0, and

$$k \equiv \sin(\alpha_3),\ \phi_i \equiv \arcsin\left[\frac{\sin(\alpha_i)}{k}\right],\ i=0,2.$$

The transcendental function (6) can be evaluated in terms of the elliptic integrals of the first kind [5, Appendix E]; [8, p. 859], but for our purposes this is not necessary, as we are looking for the minimum of (5). Let us prove that the desired minimum corresponds to $\varphi_2 = \varphi_2^{(\infty)} \equiv 2\pi - 3\varphi_3$, and consequently,

$$\left.\frac{\partial T}{\partial \alpha_2}\right|_{\alpha_2=\pi-3\alpha_3} = 0.$$

Comment. The superscript $^{(\infty)}$ indicates that starting from $\varphi_2^{(\infty)}$ the trajectory becomes a circular arc, which consists of an infinite number of chords.

To get the derivative of (6) with respect to the parameter α_2, we can apply the *Leibniz rule* [9, p. 103]:

$$\frac{d}{d\alpha_2}\int_{\phi_0}^{\phi_2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}} = \frac{1}{\sqrt{1-\sin^2(\alpha_3)\frac{\sin^2(\alpha_2)}{\sin^2(\alpha_3)}}}\cdot\frac{d\phi_2}{d\alpha_2} = \frac{1}{\cos(\alpha_2)}\cdot\frac{\cos(\alpha_2)}{\sqrt{\sin^2(\alpha_3)-\sin^2(\alpha_2)}}$$

$$= \frac{1}{\sqrt{\sin(\alpha_3+\alpha_2)\sin(\alpha_3-\alpha_2)}} \to \left.\frac{d}{d\alpha_2}\int_{\phi_0}^{\phi_2}\frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}\right|_{\alpha_2=\pi-3\alpha_3} = \frac{1}{\sqrt{-\sin(2\varphi_3)\sin\varphi_3}}.$$

The derivative of the first term in (5) becomes

$$\left[\frac{\sin(\alpha_3-\alpha_2)}{\sqrt{F_2}}\right]' = \frac{\dfrac{\sin\varphi_2}{2\sin(\alpha_3+\alpha_2)} - \cos(\alpha_3-\alpha_2)}{\sqrt{F_2}} \to \left[\frac{\sin(\alpha_3-\alpha_2)}{\sqrt{F_2}}\right]'_{\alpha_2=\pi-3\alpha_3}$$
$$= \frac{\cos(2\varphi_3)}{\sqrt{-\sin(2\varphi_3)\sin\varphi_3}} - \frac{3\sin\varphi_3 - 4\sin^3(\varphi_3)}{2\sqrt{-\sin(2\varphi_3)\sin\varphi_3}\cdot\sin\varphi_3} = \frac{\cos(2\varphi_3)}{\sqrt{-\sin(2\varphi_3)\sin\varphi_3}}$$
$$-\frac{3}{2\sqrt{-\sin(2\varphi_3)\sin\varphi_3}} + \frac{2\sin^2\varphi_3}{\sqrt{-\sin(2\varphi_3)\sin\varphi_3}} = -\frac{1}{2\sqrt{-\sin(2\varphi_3)\sin\varphi_3}}.$$

Here we took into account that

$$\sin\left[\alpha_3+\alpha_2^{(\infty)}\right] = \sin(\pi-\varphi_3) = \sin\varphi_3,\ \sin\left[\alpha_3-\alpha_2^{(\infty)}\right] = \sin(2\varphi_3-\pi) = -\sin(2\varphi_3),$$
$$\cos\left[\alpha_3-\alpha_2^{(\infty)}\right] = \cos(2\varphi_3-\pi) = -\cos(2\varphi_3),\ \sin\left[\varphi_2^{(\infty)}\right] = -\sin(3\varphi_3),$$

and applied the trigonometric identity $\sin(3\varphi_3) = 3\sin\varphi_3 - 4\sin^3(\varphi_3)$. Thus,

$$\left.\frac{\partial T}{\partial\alpha_2}\right|_{\alpha_2=\pi-3\alpha_3} = -\frac{1}{2\sqrt{-\sin(2\varphi_3)\sin\varphi_3}} + \frac{1}{2\sqrt{-\sin(2\varphi_3)\sin\varphi_3}} = 0.$$

It is easy to be convinced that $\varphi_2^{(\infty)} = 2\pi - 3\varphi_3$ is a minimum (not maximum) of (5). Indeed, we can simply check that the second derivative of the descent time is positive at $\alpha_2 = \pi - 3\alpha_3$, but the same conclusion can be drawn even more easily by calculating the first derivative at the largest possible value $\alpha_2 = \pi/2 - \alpha_3$. By doing so, we get

$$\left.\frac{\partial T}{\partial\alpha_2}\right|_{\alpha_2=\pi/2-\alpha_3} = \frac{1}{2\sqrt{-\cos\varphi_3}} + \frac{\sin\varphi_3}{\sqrt{-\cos\varphi_3}} - \frac{\sin\varphi_3}{\sqrt{-\cos\varphi_3}} = \frac{1}{2\sqrt{-\cos\varphi_3}} > 0,$$

which is only possible if $\varphi_2^{(\infty)}$ is a minimum point.

Comment. The obtained result justifies the constraint $\varphi_F \in [0, \varphi_I - 4\alpha)$ that we set. Indeed, if $0 \le \varphi_F < \varphi_2^{(\infty)}$, then the minimum of the function $T(\alpha_2)$ exists and is independent of the end point of the trajectory. An interesting nuance is that $\varphi_2^{(\infty)} = 2\pi - 3\varphi_3$ is an important parameter in another context as well. Namely, for a fixed $\varphi_3 > \pi/2$ and for φ_2 in the range $\varphi_1 < \varphi_2 \le \varphi_2^{(\infty)}$, any two-chord descent is faster than the straight one-chord descent [7]. We will discuss this surprising coincidence in Section 3.3.

2.2. The main theoretical result

Our goal is to prove that if the conditions (2) and (3) are met, then the fastest trajectory on a circle consists of a finite number of connected chords. The following lemma is helpful to achieve this goal:

Lemma. *Let 3, 2, and 1 be the points on the circumference of a circle, such that $\varphi_1 < \varphi_2 < \varphi_3$. Suppose the conditions (2) and (3) are satisfied for points 3 and 2, respectively, and points 2 and 1 are* ***arbitrarily close*** *to each other. If a particle at rest falls from point 3 to point 2 and continues to move towards point 1, then the descent along the circular arc from point 2 to point 1 is* ***slower*** *than the descent along the chord 2-1.*

Proof. For the descent time (in dimensionless units) along the circular arc from point 2 to 1 we can apply Eq. (6), replacing φ_0 with φ_1. Thus,

$$t_1 = \frac{1}{2}\int_{\phi_1}^{\phi_2} \frac{d\beta}{\sqrt{1-k^2\sin^2\beta}}. \tag{7}$$

The corresponding descent time along the chord 2-1 reads as

$$t_2 = \frac{\sin(\alpha_2-\alpha_1)}{\sqrt{F_1}+\sqrt{F_2}}, \tag{8}$$

according to (4). Let φ_2 and φ_3 be fixed, so that both t_1 and t_2 can be treated as the functions of the final angle α_1. Obviously, $t_1(\alpha_2) = t_2(\alpha_2) = 0$. To calculate the first derivative of (7), we can apply the *Leibniz rule*. Thereafter, we can easily calculate the higher derivatives of this function:

$$t_1' = \frac{dt_1(\alpha_1)}{d\alpha_1} = -\frac{1}{2\sqrt{F_1}},\ t_1'' = -\frac{\sin(\varphi_1)}{4F_1^{3/2}},\ t_1''' = -\frac{\cos(\varphi_1)}{2F_1^{3/2}} - \frac{3\sin^2(\varphi_1)}{8F_1^{5/2}},$$

$$t_1'''' = \frac{\sin(\varphi_1)}{F_1^{3/2}} - \frac{9\sin(2\varphi_1)}{8F_1^{5/2}} - \frac{15\sin^3(\varphi_1)}{16F_1^{7/2}}.$$

The derivatives of the function (8) read as

$$t_2' = \frac{dt_2(\alpha_1)}{d\alpha_1} = -\frac{\cos(\alpha_2-\alpha_1)}{\sqrt{F_1}+\sqrt{F_2}} + \frac{\sin(\alpha_2-\alpha_1)}{\left(\sqrt{F_1}+\sqrt{F_2}\right)^2}\cdot\frac{\sin(\varphi_1)}{2\sqrt{F_1}},$$

$$t_2'' = -\frac{\sin(\alpha_2-\alpha_1)}{\sqrt{F_1}+\sqrt{F_2}} - \frac{\cos(\alpha_2-\alpha_1)}{\left(\sqrt{F_1}+\sqrt{F_2}\right)^2}\cdot\frac{\sin(\varphi_1)}{\sqrt{F_1}}$$
$$+\sin(\alpha_2-\alpha_1)\cdot\left[\frac{\sin(\varphi_1)}{2\sqrt{F_1}\left(\sqrt{F_1}+\sqrt{F_2}\right)^2}\right]'.$$

General expressions for t_2''' and t_2'''' look rather complex, and they are not given here. In order to prove the assertion, let us expand both functions around α_2, so that $\alpha_1 = \alpha_2 - \Delta$. Thus, taking into account that

$$t_1'(\alpha_2) = -\frac{1}{2\sqrt{F_2}},\ t_1''(\alpha_2) = -\frac{\sin(\varphi_2)}{4F_2^{3/2}},\ t_1'''(\alpha_2) = -\frac{\cos(\varphi_2)}{2F_2^{3/2}} - \frac{3\sin^2(\varphi_2)}{8F_2^{5/2}},$$

$$t_1''''(\alpha_2) = \frac{\sin(\varphi_2)}{F_2^{3/2}} - \frac{9\sin(2\varphi_2)}{8F_2^{5/2}} - \frac{15\sin^3(\varphi_2)}{16F_2^{7/2}},$$

and correspondingly,

$$t_2'(\alpha_2) = -\frac{1}{2\sqrt{F_2}},\ t_2''(\alpha_2) = -\frac{\sin(\varphi_2)}{4F_2^{3/2}},\ t_2'''(\alpha_2) = \frac{1}{2\sqrt{F_2}} - \frac{3\cos(\varphi_2)}{4F_2^{3/2}} - \frac{3\sin^2(\varphi_2)}{8F_2^{5/2}},$$

$$t_2''''(\alpha_2) = \frac{\sin(\varphi_2)}{2F_2^{3/2}} + \frac{2\sin(\varphi_2)}{F_2^{3/2}} - \frac{\sin(2\varphi_2)}{F_2^{5/2}} + \frac{13\sin^3(\varphi_2)}{16F_2^{7/2}},$$

we get

$$t_1 = \frac{\Delta}{2\sqrt{F_2}} - \frac{\sin(\varphi_2)}{8F_2^{3/2}}\Delta^2 + \left[\frac{\cos(\varphi_2)}{12F_2^{3/2}} + \frac{\sin^2(\varphi_2)}{16F_2^{5/2}}\right]\Delta^3 + \ldots,\text{and}$$

$$t_2 = \frac{\Delta}{2\sqrt{F_2}} - \frac{\sin(\varphi_2)}{8F_2^{3/2}}\Delta^2 + \left[\frac{\cos(\varphi_2)}{8F_2^{3/2}} + \frac{\sin^2(\varphi_2)}{16F_2^{5/2}} - \frac{1}{12\sqrt{F_2}}\right]\Delta^3 + \ldots$$

We see that these expressions only differ in higher order terms. Namely,

$$\begin{aligned}\Delta t = t_2 - t_1 &= \frac{\Delta^3}{24F_2^{3/2}}\left[\cos(\varphi_2) - 2F_2\right] + \frac{\sin(\varphi_2)\cdot\Delta^4}{16F_2^{3/2}}\left[1 + \frac{\cos(\varphi_2)}{6F_2} + \frac{7\sin^2(\varphi_2)}{6F_2^2}\right] + \ldots \\ &= \frac{\cos(\varphi_3)\cdot\Delta^3}{24F_2^{3/2}} + \frac{\sin(\varphi_2)\cdot\Delta^4}{16F_2^{3/2}}\left[1 + \frac{\cos(\varphi_2)}{6F_2} + \frac{7\sin^2(\varphi_2)}{6F_2^2}\right] + O(\Delta^5).\end{aligned} \tag{9}$$

Here, we took into account that

$$\cos(\varphi_2) - 2F_2 = \cos^2(\alpha_2) - 2\sin^2(\alpha_3) + \sin^2(\alpha_2) = \cos(\varphi_3). \tag{10}$$

If $\varphi_3 > \dfrac{\pi}{2}$, then $\cos(\varphi_3) < 0$, and consequently, $t_1 > t_2$, which proves the assertion for any sufficiently small (but finite) $\Delta = \alpha_2 - \alpha_1$. An explicit expression for the upper bound of Δ can be easily deduced from (9). □

Corollary. The second term in (9) is definitely positive if $\varphi_1 < \varphi_2 \leq \varphi_3$, while the first term becomes zero if $\varphi_3 = \pi/2$. Consequently, $t_1 < t_2$ in this special case. From this, we can conclude that the descent along the circular arc from point 2 to point 1 is faster than along any broken line of chords connecting these points. Thus, as a an auxiliary result, we can confirm Galileo's conjecture for the highest possible starting point in the lower quadrant of a circle.

In view of the obtained results, we can now formulate the following theorem:

Theorem. *Suppose I is a starting point of a particle at rest on a circle, so that $\varphi_I = \pi/2 + \alpha$ and $\alpha \in (0, \pi/6)$. Then it is always possible to construct a broken line of a finite number of chords, for which the descent to the end point F, subjected to the constraint $\varphi_F \in [0, \varphi_I - 4\alpha)$, is faster than the descent along the circular arc from point I to point F.*

Proof. The proof is elementary. First, we replace the initial part of the circular trajectory with the straight chord from φ_I to $\varphi_I - 4\alpha$. Then, step-by-step, we replace the remaining circular arc with a broken line of n chords, applying the lemma we just proved. This can be done as follows. Let $\varphi_3 = \varphi_I$ be fixed and define two sets of angles:

$$\varphi_1^{(k)} = \varphi_2^{(k-1)},\ \varphi_2^{(k)} = \varphi_1^{(k)} + \Delta_k,\ \varphi_2^{(0)} = \varphi_F.$$

The increments of angles should be sufficiently small, so that we can apply the lemma. At each step of the procedure, we put $\varphi_1 = \varphi_1^{(k)}$, $\varphi_2 = \varphi_2^{(k)}$ ($k = 1, 2, \ldots, n$), and apply the lemma, i.e., we replace the arc from φ_2 to φ_1 with the corresponding chord. As a result, the overall descent time is shortened. The described procedure can be repeated as many times as needed. The final step is

$$\varphi_2^{(n)} = \varphi_1^{(n)} + \Delta_n = \varphi_F + \sum_{i=1}^{n}\Delta_i = \varphi_I - 4\alpha.$$

In this way, the initial circular arc from φ_I to φ_F can be replaced, step-by-step, with an appropriate finite set of chords. At each step the descent time is shortened. Therefore, the overall descent time is definitely shortened as well. □

It seems useful to end this theoretical section with an explicit numerical example. In Fig. 3, a comparison is made between the descent times along chord-arc (red curves) and two-chord (blue curves) trajectories. The particle is expected to be initially at rest, and the starting angles φ_I vary from 90 to 92° with step 0.2° (the lower curves correspond to smaller φ_I). The minima of the red curves are located at $\varphi_{\min} = \varphi_I - 4\alpha$ ($\alpha = \varphi_I - 90°$), and they are lower than the minima of the corresponding blue curves. In addition, the minimum descent times (NB! Not the descent curves) for the same range of starting angles along three-chord trajectories are shown (green horizontal lines). If $\varphi_I = 90°$, then the fastest descent corresponds to

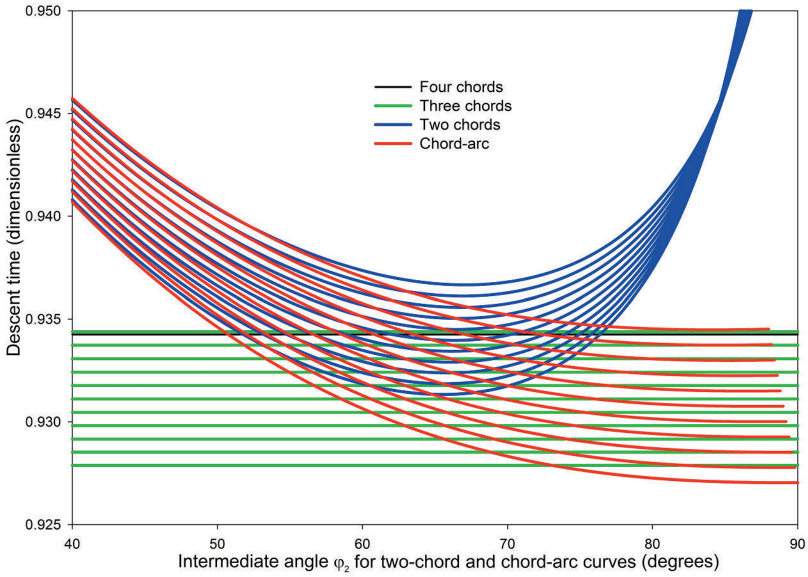

Fig. 3. Dimensionless descent times along different paths from the same starting point to the lowest point of the circle.

the circular arc itself, while for $\varphi_I = 92°$ the absolute minimum of the descent time is achieved along the four-chord path with the following set of angles: $\varphi_I = \varphi_4^{(0)} = 92°$, $\varphi_4^{(1)} = 85.747°$, $\varphi_4^{(2)} = 67.088°$, and $\varphi_4^{(3)} = 37.162°$. This is demonstrated by the black horizontal line in Fig. 3. Note also that for $\varphi_I = 92°$ (as well as for $\varphi_I = 91.8°$) the descent along the optimal three-chord trajectory is faster than along the corresponding chord-arc path. In all cases, the end point is the lowest point of the circle ($\varphi_F = 0$). The details of these computations will be explained in the subsequent sections.

3. GENERAL APPROACH TO THE PROBLEM

We have proved that the fastest descent for a particle at rest on a circle, from any starting angle $\varphi_I > 90°$ to a fixed end angle $\varphi_F \in [0, \varphi_I - 4\alpha)$, is achieved along some broken line of a finite number of chords. In this connection, a natural question arises: can we ascertain the optimal number of chords and their positions? The problem may seem too big to handle, and indeed, this is a serious challenge. Nevertheless, our goal is to work out an extrapolation scheme, which, in principle, would enable to determine the optimal n-chord broken line for any starting angle $\varphi_I \in (\pi/2, 2\pi/3)$. Such an ambitious goal can only be achieved step-by-step. We therefore begin with the analysis of the most simple cases and then try to extend the approach to more complex situations.

3.1. One chord vs two chords

Suppose $\varphi_3 < \dfrac{2\pi}{3}$ and let us compare descents for one- and two-chord paths, both starting at rest at φ_3 and ending at $\varphi_0 = 0$. The corresponding descent times are

$$T_1 = t_{30}^{(3)} = \frac{\sin(\alpha_3) - \sqrt{\sin^2(\alpha_3) - \sin^2(\alpha_3)}}{\sin(\alpha_3)} = 1 \text{ – Galileo's result!}$$
$$T_2 \equiv f(\alpha_2) = t_{32}^{(3)} + t_{20}^{(3)} = \frac{\sin(\alpha_3 - \alpha_2)}{\sqrt{F_2}} + \frac{\sin(\alpha_3) - \sqrt{F_2}}{\sin(\alpha_2)} \tag{11}$$
$$= \frac{\sin(\alpha_3 - \alpha_2)}{\sqrt{F_2}} + \frac{\sin(\alpha_2)}{\sin(\alpha_3) + \sqrt{F_2}} \equiv f(\alpha_2) \to f(0) = 1.$$

Here, we took into account that
$$\sin(\alpha_3) - \sqrt{F_2} = \frac{\sin^2(\alpha_2)}{\sin(\alpha_3) + \sqrt{F_2}}.$$

The derivative of $f(\alpha_2)$ reads as

$$f'(\alpha_2) = -\frac{\cos(\alpha_3 - \alpha_2)}{\sqrt{F_2}} + \frac{\sin(\alpha_2)\cos(\alpha_2)}{\sin(\alpha_3 + \alpha_2)\sqrt{F_2}} + \frac{\sin(\alpha_3)\cos(\alpha_2)}{\left[\sin(\alpha_3) + \sqrt{F_2}\right]\sqrt{F_2}}. \tag{12}$$

Therefore,
$$f'(0) = -\frac{\cos(\alpha_3)}{\sin(\alpha_3)} + \frac{1}{2\sin(\alpha_3)} = \frac{1/2 - \cos(\alpha_3)}{\sin(\alpha_3)} < 0,$$

as $\varphi_3 < \frac{2\pi}{3}$, and consequently, $\cos(\alpha_3) > 1/2$. It means that for any $\varphi_3 < \frac{2\pi}{3}$ an angle α_2 exists, so that $f(\alpha_2) = f(0) + f'(0)\,\alpha_2 < 1$. Indeed, the function $f(\alpha_2)$ and its derivative $f'(\alpha_2)$ are both continuous, while $f(0) = 1$ and also $f(\delta_1) = 1$, where, according to [7],

$$\delta_1 = \arccos\left\{\frac{3 - \cos(\alpha_3) - 4\cos^2(\alpha_3) + [1 - \cos(\alpha_3)]\,[4\cos(\alpha_3) - 1]\sqrt{9 - 16\cos(\alpha_3)}}{2\,[5 - 12\cos(\alpha_3) + 8\cos^2(\alpha_3)]}\right\}.$$

Thus, according to *Rolle's theorem* [10], at least one value $\alpha_2 \in (0, \delta_1)$ exists, such that $f'(\alpha_2) = 0$. Moreover, as $f'(0) < 0$, a minimum $\alpha_{\min}$ exists in this range, where $f(\alpha_{\min}) < 1$. It does not mean that any two-chord path is faster than the straight one-chord path if $\varphi_3 < \frac{2\pi}{3}$ (as has been proved in [7]; this would be the case if $\varphi_3 < 2\delta_0 = 2\arccos(9/16) \approx 111.54°$). However, for any $\alpha_3 < \frac{\pi}{3}$ an appropriate $\alpha_2 < \alpha_3$ can be found, for which $T_2 < T_1 = 1$.

3.2. A more general analysis of the two-chord descent

In order to demonstrate some important specific features of the two-chord descent, let us temporarily abandon the constraint (1). A motivation for this more general approach can be seen in Figs 4 and 5, where several calculated two-chord descent curves are shown.

Quite surprisingly, for any starting angle φ_3 above a critical value $\theta \approx 103.65°$, these curves have two minima with exactly the same descent time! Of course, there is a maximum $\varphi_2^{\max}$ between the two minima, while the descent time $f(\varphi_2^{\max}) > 1$ if $\varphi_3 > 2\delta_0 \approx 111.54°$, and $f(\varphi_2^{\max}) < 1$ otherwise. At $\varphi_3 = \theta \approx 103.65°$ the three extremum points coincide, and for any $\varphi_3 \leq \theta$ there remains only a single minimum in the "physical" region $\varphi_2 \leq \pi - \varphi_3$.

Moreover, the boundary value $\pi - \theta \approx 76.35°$ for φ_2 exactly coincides with the limit where three extremum points of the descent curve transform into a single minimum. This means that the first step of the descent is a free fall from the angle θ to $\pi - \theta$ (see Fig. 5). The described results of a careful numerical analysis reveal some new interesting aspects of the problem. It would therefore be desirable to rigorously prove the correctness of these results. The proof is given in Appendix C (see [3]).

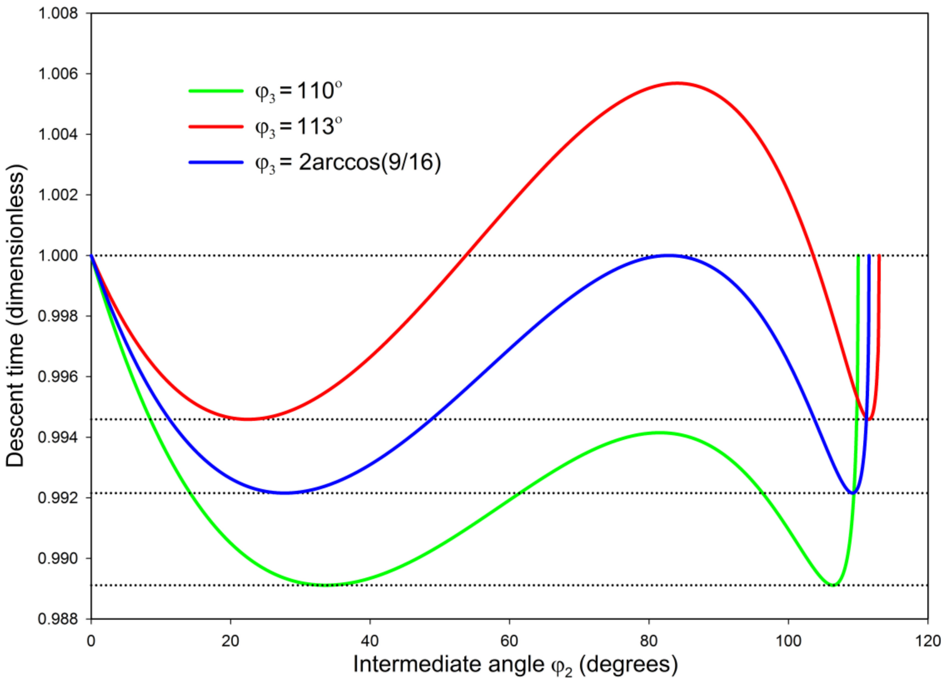

Fig. 4. A demonstration of the specific features of two-chord descent curves. Note that $2\delta_0 = 2\arccos(9/16) \approx 111.54°$ is the upper bound of the region where any two-chord path is faster than the straight one-chord path [7].

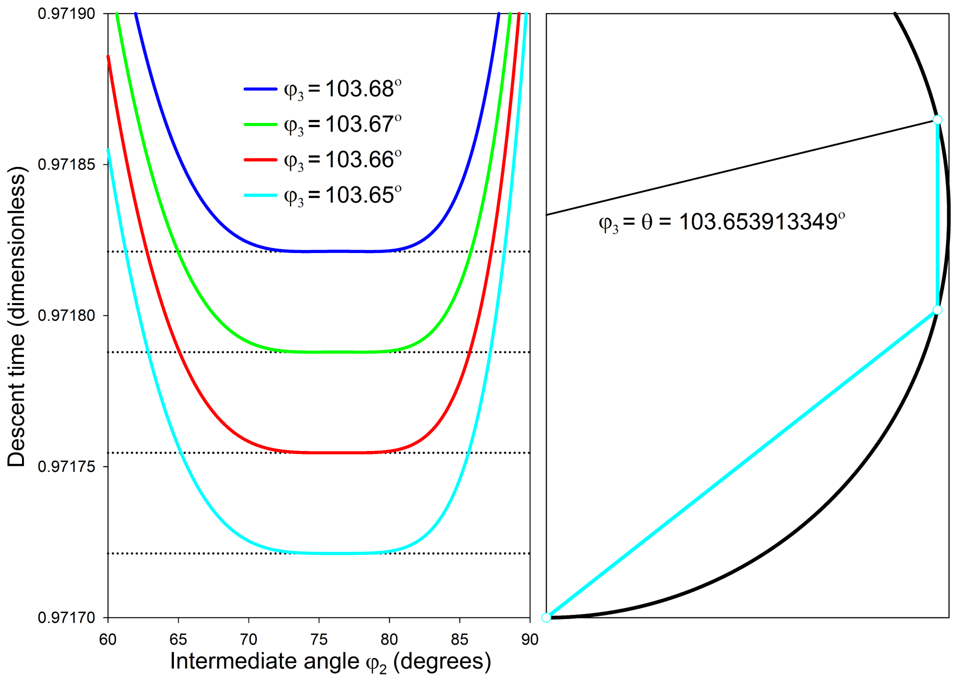

Fig. 5. Locating the critical starting angle θ for two-chord descent curves. As θ can be considered an important physical-geometrical parameter of a circle, it has been determined with high accuracy.

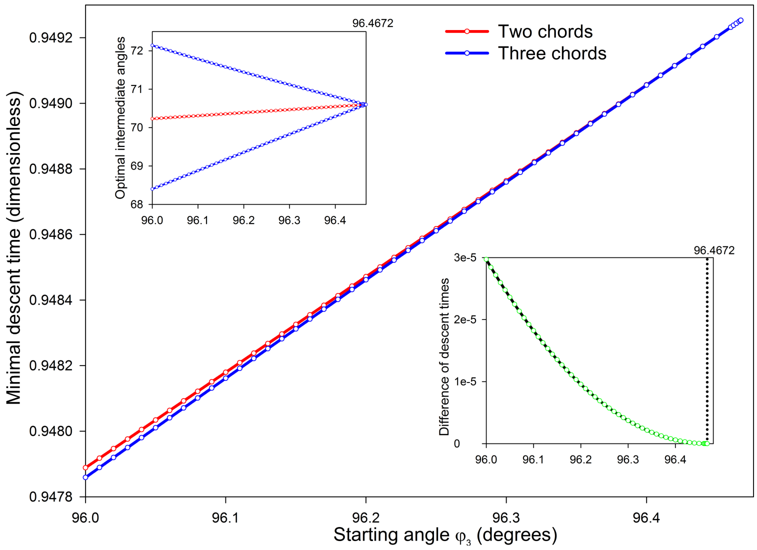

Fig. 6. Locating the values of $\varphi_2^{(0)}$ for an optimal descent along two- and three-chord paths.

3.3. The lower bound of the two-chord region

In Section 3.1 we proved that when a starting angle φ_I is slightly smaller than 120° and $\varphi_F = 0°$, then the fastest descent is achieved along an appropriate two-chord path. How could we determine the lower bound of the region where the two-chord descent is the fastest? To this end, we have to make a comparison between the descent times of a three-chord path S_3 and a two-chord path S_2 from the same starting angle φ_3 to the same final angle $\varphi_0 = 0$. Let us assume that the intermediate angle φ_2 is fixed as well. The difference between the descent times of S_3 and S_2 is then a function of a single argument $\alpha_1 \in [0, \alpha_2]$:

$$F(\alpha_1) \equiv t_{21}^{(3)} + t_{10}^{(3)} - t_{20}^{(3)},\ F(\alpha_2) = 0,\ F(0) = 0,\ \alpha_i = \varphi_i/2\ (i = 1, 2).$$

To some surprise, the optimal lower intermediate angle $\varphi_3^{(2)} = 2\alpha_2^{(2)}$ for any S_3 can be ascertained analytically. Namely,

$$\cos\left[\alpha_2^{(2)}\right] = \frac{1}{1 - 2\cos(\varphi_3)}.$$

The optimal value of the higher intermediate angle $\varphi_3^{(1)}$ for S_3 can also be determined in a straightforward manner, by solving numerically a relatively simple equation (see [3], Appendix D, for details).

In this way, we can also determine the boundary value for $\varphi_2^{(0)}$, below which the optimal three-chord descent becomes faster than any two-chord descent. Indeed, in the immediate vicinity of this critical point, the dependence of both optimal intermediate angles (φ_2 and φ_1) on the starting angle φ_3 is nearly linear. The desired result, $\varphi_2^{(0)} = 96.4672°$, corresponds to the crossing of these nearly linear curves, as demonstrated in Fig. 6. In addition, the optimal value $\varphi_2^{(1)} = 70.5982°$ was determined. Consequently, for any starting angle in the range $\varphi_3 \in (96.4672°, 120°)$, the fastest descent is achieved along an appropriate two-chord path. Note that $\varphi_2^{(1)} + 3\varphi_2^{(0)} = 360°$ (exactly!). This is the somewhat surprising coincidence mentioned at the end of Section 2.1. Namely, if $\varphi_3 = \varphi_2^{(0)}$, then $\varphi_2^{(1)} = \varphi_2^{(\infty)}$.

3.4. General formula for the descent time

Our main goal in this paper is to work out an extrapolation scheme, which, in principle, would enable to determine the optimal n-chord path for any starting angle $\varphi \in (\pi/2, 2\pi/3)$. The first steps towards this goal were described in the previous three subsections, but it is not at all clear how to extend the analysis to more complex situations. However, the basis for the analysis is quite clear. Indeed, to calculate the descent time for an n-chord broken line ($\varphi_n > \varphi_{n-1} > ... > \varphi_0$), we can use the following general formula:

$$t_n(\varphi_n) = f(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_{n-1}) \equiv t_{n,n-1}^{(n)} + t_{n-1,n-2}^{(n)} + ... + t_{10}^{(n)} + t_0^{(n)}, \tag{13}$$

$$t_{n,n-1}^{(n)} = \frac{\sin(\alpha_n - \alpha_{n-1})}{\sqrt{F_{n-1}}},\ t_{k,k-1}^{(n)} = \frac{\sin(\alpha_k - \alpha_{k-1})}{\sqrt{F_k} + \sqrt{F_{k-1}}},\ t_0^{(n)} = \frac{\sin(\alpha_n) - \sqrt{F_0}}{\sin(\alpha_0)},$$

$$\alpha_k \equiv \varphi_k/2,\ F_k \equiv \sin^2(\alpha_n) - \sin^2(\alpha_k) = \sin(\alpha_n + \alpha_k)\sin(\alpha_3 - \alpha_k),\ k = 1, 2, ..., n.$$

The corresponding partial derivatives read as

$$\frac{\partial f}{\partial \alpha_0} = -\frac{\cos(\alpha_1 - \alpha_0)}{\sqrt{F_0} + \sqrt{F_1}} + \frac{\sin(\alpha_1 - \alpha_0)}{\left(\sqrt{F_0} + \sqrt{F_1}\right)^2} \cdot \frac{\sin(\alpha_0)\cos(\alpha_0)}{\sqrt{F_0}} - \frac{\left[\sin(\alpha_n) - \sqrt{F_0}\right]\cos(\alpha_0)}{\sin^2(\alpha_0)} + \frac{\cos(\alpha_0)}{\sqrt{F_0}}, \tag{14}$$

$$\frac{\partial f}{\partial \alpha_k} = -\frac{\cos(\alpha_{k+1} - \alpha_k)}{\sqrt{F_k} + \sqrt{F_{k+1}}} + \frac{\sin(\alpha_{k+1} - \alpha_k)}{\left(\sqrt{F_k} + \sqrt{F_{k+1}}\right)^2} \cdot \frac{\sin(\alpha_k)\cos(\alpha_k)}{\sqrt{F_k}} + \frac{\cos(\alpha_k - \alpha_{k-1})}{\sqrt{F_{k-1}} + \sqrt{F_k}} + \frac{\sin(\alpha_k - \alpha_{k-1})}{\left(\sqrt{F_{k-1}} + \sqrt{F_k}\right)^2} \cdot \frac{\sin(\alpha_k)\cos(\alpha_k)}{\sqrt{F_k}},\ k = 1, 2, ..., n-2, \tag{15}$$

$$\frac{\partial f}{\partial \alpha_{n-1}} = -\frac{\cos(\alpha_n - \alpha_{n-1})}{\sqrt{F_{n-1}}} + \frac{\sin(\alpha_n - \alpha_{n-1})}{\sqrt{F_{n-1}^3}}\sin(\alpha_{n-1})\cos(\alpha_{n-1}) + \frac{\cos(\alpha_{n-1} - \alpha_{n-2})}{\sqrt{F_{n-1}} + \sqrt{F_{n-2}}} + \frac{\sin(_{n-1} - \alpha_{n-2})}{\left(\sqrt{F_{n-1}} + \sqrt{F_{n-2}}\right)^2} \cdot \frac{\sin(\alpha_{n-1})\cos(\alpha_{n-1})}{\sqrt{F_{n-1}}}. \tag{16}$$

If the starting angle φ_n is fixed, then the absolute minimum of the descent time can only be at a point where $\mathbf{grad} f(\alpha_0, \alpha_1, \alpha_2, ..., \alpha_{n-1}) = 0$, i.e., all partial derivatives, (14)–(16), should vanish (see, e.g., [11], p. 744). Of course, this necessary condition also applies to the two- and three-chord descent times analyzed in the previous subsection: the corresponding gradients indeed vanish in all cases studied there. A typical three-chord descent curve in the immediate vicinity of its minimum is shown in Fig. 7. The curve is located on the plane $\varphi_2 - \varphi_1 = 33.6454°$, and both partial derivatives (at $\varphi_2 = 83.6206°$ and $\varphi_1 = 49.9752°$) are zero to better than 10^{-8} accuracy. One might think that $t_{\min}$ is given to too many decimal places, but have a look at the scale of the z-axis. You see that even the highest points of the curve differ from the minimum value by less than $3 \cdot 10^{-13}$ units. Moreover, to accurately locate the minimum point, much higher precision is needed. Thus, $t_{\min} = 0.9446336101255117$, where none of the digits is meaningless. One should not think that, e.g., $t_{\min} = 0.94463361$ would have been sufficiently good for our purposes.

Determining the two- and three-chord descent times on a circle is quite simple, as we can apply the analytical results described in the previous sections. The analysis of the four- and five-chord descent is more complex but still feasible in a straightforward manner using, e.g., the downhill simplex method [12]. The details of this analysis as well as the obtained results are described in Appendix D (see [3]).

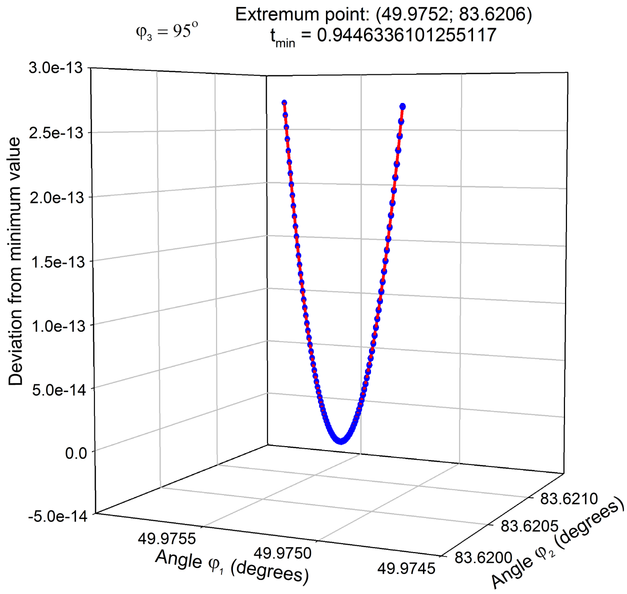

Fig. 7. A demonstration of a three-chord descent curve near its absolute minimum.

4. EXTRAPOLATION SCHEME

We have described how to determine the fastest descent time for a broken line of $n = 2,3,4$, and 5 chords on a circle (see [3], Appendix D), and fixed the regions where the optimal number of chords is $n = 2,3$, or 4. One might expect that further computations become more and more complex, but fortunately, this is not quite the case, as can be inferred from Fig. 8. The point is that the already obtained results can be successfully used to get a reliable initial guess for the next step. Moreover, by increasing the number of chords, the initial guesses become more and more precise! Thus, the absolute minima of the corresponding functions of a large number of variables can still be located quite easily.

Let us explain this in more detail. All dots in Fig. 8 are related to the boundary values of the regions where minimal descent times for the n-chord and $n+1$-chord paths become equal. Note that starting from $n = 3$, two sets of dots with the same color are shown. They correspond to the same starting angle $\varphi_n^{(0)}$, but to different number of chords: n (larger dots) or $n+1$ (smaller dots). As can be seen, the positions of these dots are not at all arbitrary: they can be joined by smooth curves. Of course, this is only a methodological trick, as the number of chords is an integer by definition. Nevertheless, the trick is very useful, as it can easily be put into practice. Note, however, that only the second set of starting angles (shown by smaller dots) can be used for extrapolation.

Suppose we have determined all characteristic angles for the optimal descent along an n-chord path, and the corresponding parameters for any smaller number of chords are known as well. Then a reliable initial guess for the new boundary value $\varphi_{n+1}^{(0)}$, and for any intermediate angle $\varphi_{n+1}^{(k)}$ $(k = 1,2,...,n-1)$ can be obtained by fitting the already known values $\varphi_n^{(k)}$, $\varphi_{n-1}^{(k)}$, $\varphi_{n-2}^{(k)}$, and $\varphi_{n-3}^{(k)}$ $(k = 0,1,2,...,n-1)$ to a simple smooth function, e.g., to $f(x) = a + b/x + c/x^2 + d/x^3$, where $x_i = n - i$ and $f(x_i) = \varphi_{n-i}^{(k)}$, $i = 0,1,2,3$ (the superscript in parentheses enumerates the angles in descending order). As there are four equations and four unknowns, the parameters a,b,c and d can be fixed exactly. The desired initial guess for $\varphi_{n+1}^{(k)}$ is then obtained simply by extrapolation:

$$\varphi_{n+1}^{(k)} = a + \frac{b}{n+1} + \frac{c}{(n+1)^2} + \frac{d}{(n+1)^3}, \; k = 0,1,2,...,n-1.$$

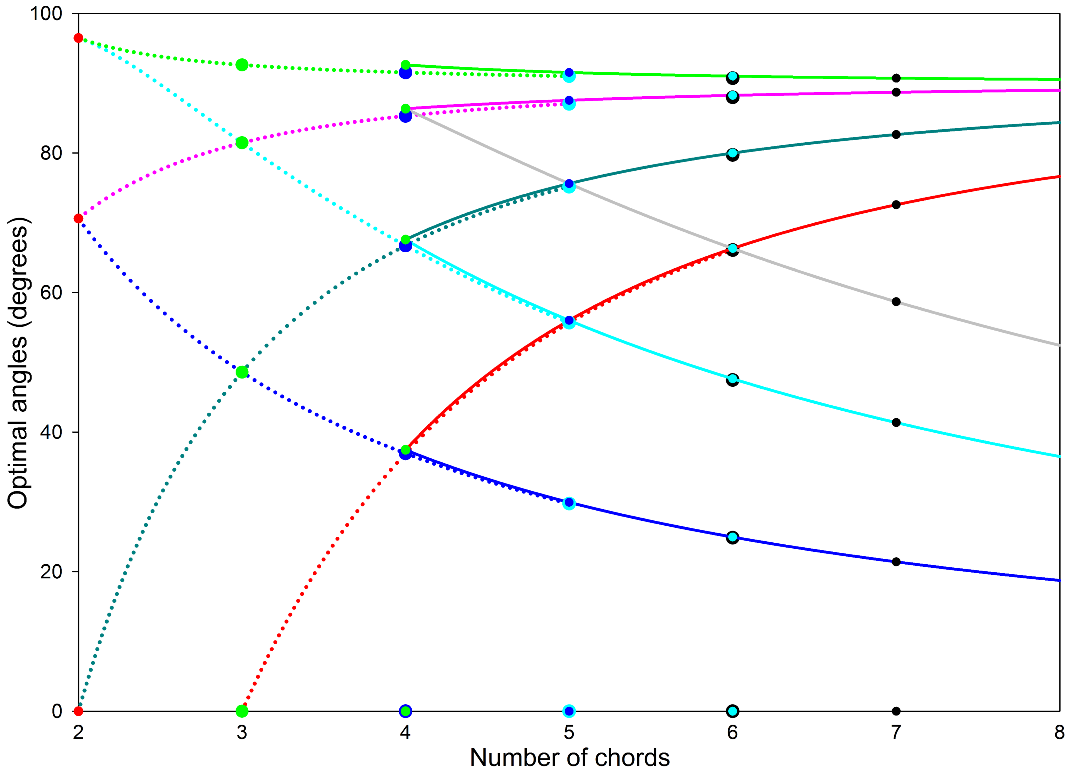

Fig. 8. A graphical presentation of the elaborated extrapolation scheme. The exact positions of the dots are given in Table 1 (see also explanations in the text).

The described simple scheme works nicely, so there is no need to apply any more sophisticated approach. For example, using the data available for an n-chord descent up to $n = 7$, we get the following guesses for $n = 8$ (upper bound):

$$\varphi_7^{(0)} = 90.534^\circ,\ \varphi_7^{(1)} = 88.991^\circ,\ \varphi_7^{(2)} = 84.359^\circ,$$
$$\varphi_7^{(3)} = 76.642^\circ,\ \varphi_7^{(4)} = 52.426^\circ,\ \varphi_7^{(5)} = 36.498^\circ,\ \varphi_7^{(6)} = 18.740^\circ.$$

These values correspond to the end points of the curves shown in Fig. 8 (not indicated by dots, as they are not the computed values). For any starting angle $\varphi < \varphi_7^{(0)} \approx 90.534^\circ$, the path for the fastest descent therefore consists of at least eight chords. Determining the exact values of $\varphi_7^{(k)}$ $(k = 0, 1, 2, ..., 6)$ is not too complicated, but one has to stop at some point. If the starting angle approaches $\pi/2$, the optimal number of chords will rapidly increase, while the descent time is already very close to the limit $t_\infty = 0.92703734$, which corresponds to the infinite number of chords, i.e., to the circular arc itself (from $\pi/2$ to 0). Thus, from the practical point of view, it does not make much sense to solve the problem for a very large (but finite) number of chords. The characteristic angles related to the minima of these n-chord $(n = 2 \div 7)$ descent curves are given in Table 1 (in degrees). They are also shown by dots in Fig. 8. All partial derivatives at $\varphi_n^{(k)}$ $(k = 1, 2, ..., n-1)$ are zero to better than 10^{-8} accuracy, which confirms the correctness of these parameters. We have also determined the starting angles for the regions where the corresponding n-chord descent becomes the fastest:

$$n = 2\colon\ \varphi \in (96.4672^\circ, 120^\circ);\ n = 3\colon\ \varphi \in (92.6179^\circ, 96.4672^\circ);$$
$$n = 4\colon\ \varphi \in (91.5326^\circ, 92.6179^\circ);\ n = 5\colon\ \varphi \in (91.0093^\circ, 91.5326^\circ);$$
$$n = 6\colon\ \varphi \in (90.7159^\circ, 91.0093^\circ);\ n = 7\colon\ \varphi \in (90.534^\circ, 90.7159^\circ).$$

Figure 9 gives an idea of these regions. Let us imagine that for any starting point on the yellow circle, we have to find the optimal path (a broken line of chords or the circular arc) which guarantees the fastest descent to the lowest point. We see that there is a wide region (shown as light blue), $\varphi \in [120°, 240°]$, where the straight one-chord descent is the fastest. An even wider region exists, $\varphi \in [-90°, 90°]$ (green semicircle), where the fastest descent is achieved along the circle itself. In contrast to this, there are two narrow sectors (shown as white in Fig. 9), $\varphi \in (90°, 90.7159°)$ and $\varphi \in (-90.7159°, -90°)$, where it is almost impossible to separate the optimal starting angles for different n-chord trajectories ($n > 7$, see Table 1).

Table 1. Characteristic angles (the dots in Fig. 8) related to the minima of the descent curves

n	$\varphi_n^{(0)}$	$\varphi_n^{(1)}$	$\varphi_n^{(2)}$	$\varphi_n^{(3)}$	$\varphi_n^{(4)}$	$\varphi_n^{(5)}$	$\varphi_n^{(6)}$
2	96.4672	70.5982					
3	92.6179	81.451	48.579				
4	92.6179	86.334	67.571	37.444			
4	91.5326	85.303	66.724	36.949			
5	91.5326	87.545	75.606	56.014	29.938		
5	91.0093	87.038	75.154	55.664	29.744		
6	91.0093	88.251	79.986	66.314	47.641	24.959	
6	90.7159	87.964	79.717	66.078	47.465	24.864	
7	90.7159	88.695	82.635	72.575	58.679	41.361	21.406

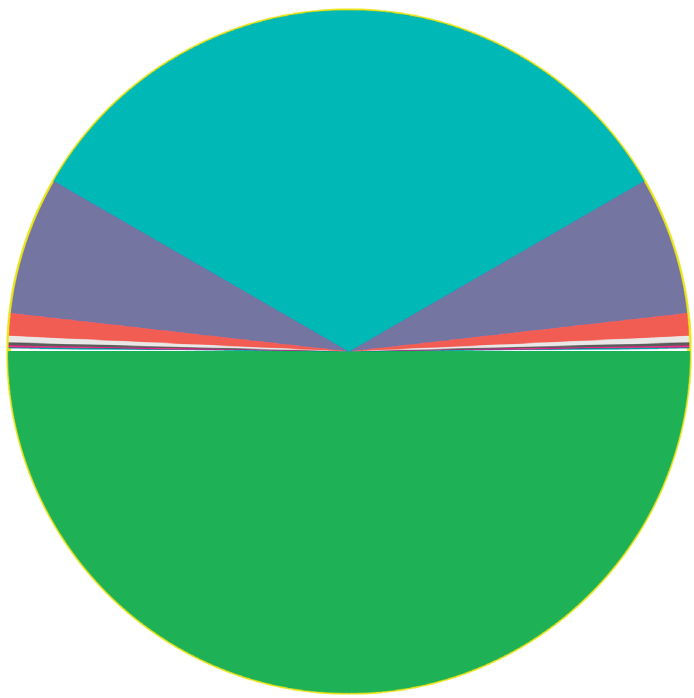

Fig. 9. Different colors highlight the regions where the fastest descent is achieved along an appropriate n-chord path: $n = 1, 2, \ldots$ The lower semicircle (shown as green) corresponds to the descent via the circular arc itself ($n = \infty$).

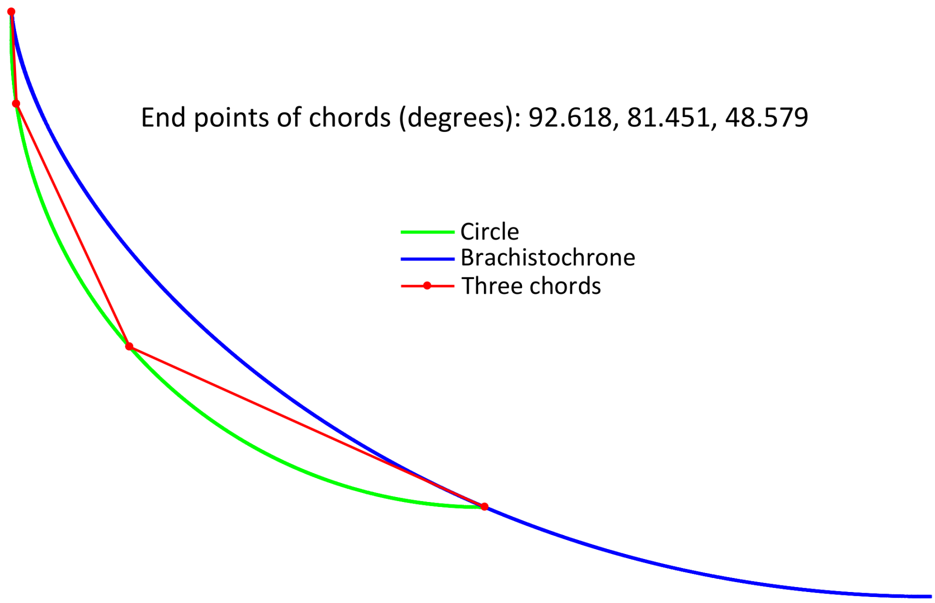

Fig. 10. A brachistochrone in comparison with the optimal three-chord path.

5. BRACHISTOCHRONE VS BROKEN LINE OF CHORDS

Galileo did not pose and certainly did not try to solve the brachistochrone problem. Instead, he discovered some very important features of the descent on a circle or connected chords on a circle. In this context, to complete our study, it would be of interest to compare the true minimum-descent-time curve (which is a brachistochrone) with an optimal n-chord trajectory on a circle. Let φ_n be the starting angle of an n-chord broken line on a circle with radius L. As previously, we assume that the end point of this path is the lowest point of the circle ($\varphi_0 = 0$). Suppose these two points of the circle are joined by the minimum-descent-time curve, so that the origin of the coordinate system corresponds to φ_n. Thus, the parametric equations of this brachistochrone are as follows (see, e.g., [5], p. 217):

$$x = a(\beta - \sin\beta),\ y = a(1 - \cos\beta),$$

where β and a are some parameters, and y-axis is directed downwards. The coordinates of the lowest point of the circle are $x_0 = L\sin\varphi_n$ and $x_0 = L(1 - \cos\varphi_n)$. Consequently, the parameters β and a can be determined by solving the equations

$$a \cdot (\beta - \sin\beta) = L\sin(\varphi_n) = 2L\sin^2(\alpha_n),$$
$$a \cdot (1 - \cos\beta) = L[1 - \cos(\varphi_n)] = 2L\sin(\alpha_n)\cos(\alpha_n) \to$$
$$\frac{1 - \cos\beta}{\beta - \sin\beta} = \tan(\alpha_n),\ \alpha_n \equiv \frac{\varphi_n}{2} \to$$
$$(\beta - \sin\beta) \cdot \tan(\alpha_n) - 1 + \cos\beta = 0. \qquad (17)$$

Equation (17) can be easily solved numerically. Thereafter, we can fix a:

$$a = L \cdot \frac{1 - \cos(\varphi_n)}{1 - \cos\beta} \to C \equiv \sqrt{\frac{a}{L}} = \frac{\sin(\alpha_n)}{\sin(\beta/2)},$$

and finally, the true minimum descent time (i.e., the descent time for the brachistochrone):

$$T_b = \beta\sqrt{\frac{a}{g}} = \frac{C\beta T_0}{2},$$

where $T_0 = 2\sqrt{L/g}$. Thus, having found the parameters β and a, the brachistochrone itself has been fixed as well.

For example, let us determine the true minimum descent curve from the starting angle $\varphi_3^{(0)} = 92.6179°$ to the lowest point of a circle (see Fig. 10). As we know, $\varphi_3^{(0)}$ is the upper bound of the region where the three-chord descent on a circle is the fastest. So, we can compare the true minimum descent time T_b with the descent time T_c for the optimal three-chord path. The result is as follows: $T_b = 0.918474556753$ ($T_0 = 1$), $T_c = 0.936460655120$, and $T_c/T_b = 1.01958$. We see that the difference is less than 2%. The fastest chord-arc descent time for the same starting angle is not much longer: $t_{chord-arc} = 0.9367923$ (achieved with $\varphi_3^{(1)} = 82.146°$), and neither is the descent time via the circular arc itself (from $\varphi_3^{(0)} = 92.6179°$ to the lowest point): $t_{circle} = 0.936961$. The obtained results may seem surprising, but they may be of practical value for modeling the real path optimization problems. This kind of studies have been of interest, e.g., to accurately model the Tour de France bicycle race [13] or to find a solution to an important practical problem – which tanks empty most rapidly [14]. On the other hand, the results of the present study have some common features with recent innovations in studying a discrete brachistochrone with an arbitrary number of segments [15].

6. CONCLUSION

We proved that Galileo's conjecture, formulated in the Scholium to Proposition 36 in his *Discorsi*, cannot be extended to a starting point in the upper half of a circle. In other words, if $\varphi_I > 90°$ and $\varphi_F = 0$, then the descent along an appropriate n-chord path ($1 \leq n < \infty$) is faster than along the circular arc from φ_I to φ_F. In contrast to this, Proposition 36 itself can be extended up to $\varphi_I = 2\arccos(9/16) \approx 111.54°$ [7], but this interesting result is of secondary importance for Galileo's conjecture.

The most important practical result of the work is the extrapolation method described in Section 4. Using this method, we determined the boundaries for the regions where the corresponding n-chord descent becomes the fastest (see Table 1 and Fig. 8). In principle, this extrapolation scheme can be applied to an arbitrarily large number of connected chords, but we only examined the n-chord trajectories up to $n = 7$. (In fact, the initial guess for the optimal eight-chord path was obtained as well.)

To avoid misunderstanding, it is necessary to explain what was **not done** in this article. For practical applications, we fixed $\varphi_F = \varphi_0 = 0$, i.e., the problem was not solved for $\varphi_0 \neq 0$. However, the main theoretical result ($n < \infty$) still remains true, and one may expect that the principles of the extrapolation method remain the same, too.

As concerns the case $\varphi_0 = 0$, the problem was not solved for **any** starting angle $\varphi_I \equiv \varphi_I^{(0)} \in (90°, 120°)$. Indeed, only the bounds for the regions of the starting angle have been determined. However, if $\varphi_n^{(0)} < \varphi_I^{(0)} < \varphi_{n-1}^{(0)}$ (see Table 1 for notations), then the number of chords of the optimal trajectory is n. This is a useful bit of information which enables to determine all intermediate angles $\varphi_I^{(k)}$ ($k = 1, 2, ..., n-1$) quite easily.

ACKNOWLEDGMENTS

I would like to thank the Publishing Board of the University of Tartu for the financial support. I am most grateful to the reviewers for their careful work and useful suggestions that helped improve the quality of this article. The publication costs of this article were partially covered by the Estonian Academy of Sciences.

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Kiireima laskumise probleem: üldine käsitlus suvalise lähtepunkti korral

Matti Selg

Artiklis analüüsitakse huvitavat mehaanikaülesannet, mille Galileo Galilei sõnastas umbes 400 aastat tagasi. Olgu I ja F kaks suvalist punkti silinderpinnal ning punktmass paigutatud punkti I, mis asub kõrgemal kui F, nii et $\varphi_F < \varphi_I$. Oletame, et punktide I ja F vahele on paigutatud n omavahel ühendatud kaldpinda, mis kõik toetuvad samale silinderpinnale. Eesmärk on leida mistahes I ja F korral optimaalne kaldpindade koguarv, nii et punktist I paigalseisust liikumist alustav punktmass jõuaks lühima ajaga punkti F. Galilei ise uuris ainult erijuhtu, mil I asub silinderpinna alumisel poolel ($\varphi_I \leq 90°$) ning F on selle pinna madalaim punkt ($\varphi_F = 0°$). Ta väitis, et libisemine mööda silinderpinda tagabki lühima langemisaja ehk teisisõnu õige vastus sellel erijuhul on $n = \infty$. Galileil oli õigus: selle väite range tõestuse võib leida artiklist [6]. Käesolev töö on Galilei probleemi üldistus: siin eeldatakse, et lähtepunkt I asub silinderpinna ülemisel poolel ($\varphi_I > 90°$). Sel juhul, nagu artiklis rangelt tõestatakse, on optimaalse trajektoori komponentide koguarv alati lõplik ($n < \infty$), kusjuures $n = 1$, kui $\varphi_I \geq 120°$ [7], ja $n \to \infty$, kui $\varphi_I \to 90°$. Lisaks nimetatud teoreetilistele tulemustele on ka välja töötatud ekstrapolatsioonimeetod, mis võimaldab kindlaks teha optimaalse trajektoori komponentide koguarvu (ja muud parameetrid) mistahes lähtenurga φ_I korral.