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TOPOLOGICAL ALGEBRAS

On the joint continuity of module multiplication

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Abstract. Let A be any topological algebra over $\mathbb{R}$ or $\mathbb{C}$. We show that the property of a topological left (right or two-sided) A-module to have a jointly continuous action of A is inherited by submodules, quotient modules, completion, direct products, direct sums, projective limits and injective limits. In the case of commutative topological A-bimodules, the same property is inherited by topological tensor products.

Keywords: joint continuity of the module multiplication, topological modules, submodules, completion, unitization, direct products and sums, projective and inductive limits, tensor product.

1. INTRODUCTION

Let $\mathbb{K}$ denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers.

Let $(A, +_A, \cdot_A, \cdot_{\mathbb{K}})$ be an algebra over $\mathbb{K}$. A linear space $(E, +_E, {}_{\mathbb{K}}\cdot)$ over $\mathbb{K}$ is a *left A-module* if there is defined an operation $\cdot : A \times E \to E$, which is called left *module multiplication* or *left action of A on E* and satisfies the properties $a \cdot (x +_E y) = a \cdot x +_E a \cdot y$, $(a +_A b) \cdot x = a \cdot x +_E b \cdot x, (a \cdot_A b) \cdot x = a \cdot (b \cdot x)$, $(\lambda \cdot_{\mathbb{K}} a) \cdot x = \lambda_{\mathbb{K}} \cdot (a \cdot x) = a \cdot (\lambda_{\mathbb{K}} \cdot x)$ for all $a, b \in A, x, y \in E, \lambda \in \mathbb{K}$. The definitions of a right A-module and a two-sided A-module are similar. For brevity of terminology and because we will present proofs only for the left-sided case, in what follows, we will refer to action instead of left action, right action or two-sided action. The only exception will be in Section 8, where we could obtain the result only for two-sided A-modules and two-sided action.

Throughout the paper, a *topological algebra* is an algebra A over the field $\mathbb{K}$, which is equipped with a topology τ such that (A, τ) is a topological linear space and the algebra multiplication is separately continuous, i.e., where the maps $l_a : A \to A$ and $r_a : A \to A$, which are defined by $l_a(b) = ab, r_a(b) = ba$ for all $b \in A$, are continuous with respect to the topology τ for each $a \in A$.

Let $(A, \tau) = (A, +_A, \cdot_A, \cdot_{\mathbb{K}}, \tau_A)$ be a topological algebra. A topological left A-module is a topological linear space $(E, \tau_E) = (E, +_E, {}_{\mathbb{K}}\cdot, \tau_E)$ over $\mathbb{K}$, which is also a left A-module, in which the (left) module multiplication $\cdot$ is separately continuous, i.e., the maps $\cdot_a : E \to E$, which are defined by $\cdot_a(x) = a \cdot x$ for all $x \in E$, are continuous for each $a \in A$ with respect to the topology τ_E. In case the map $\cdot$ is continuous, we say that (E, τ_E) is *a topological left A-module with jointly continuous action* or that *the (left) action of A on (E, τ_E) is jointly continuous.*

In what follows, we will omit the subindices of the symbols for algebraic operations for the sake of conciseness, because it is clear in which structure a particular algebraic operation is defined.

Since (E,τ_E) is a linear topological space as well, the condition of being topological left A-module with jointly continuous action could also be restated in terms of neighbourhoods of zero as follows. Let $\mathscr{N}_0$ be any base of neighbourhoods of zero in A and $\mathscr{W}_0$ be any base of neighbourhoods of zero in E. Then the map $\cdot$ is continuous if and only if, for each $W \in \mathscr{W}_0$, there exist $V \in \mathscr{N}_0$ and $U \in \mathscr{W}_0$ such that $V \cdot U = \{v \cdot u : v \in V, u \in U\} \subseteq W$. If one can show that this condition holds for only one pair $\mathscr{N}_0, \mathscr{W}_0$ of bases of neighbourhoods, then it holds for any pair $\mathscr{N}_0, \mathscr{W}_0$ of bases of neighbourhoods of zero.

Remark 1. *Notice that our definition of $V \cdot U$ differs from the similar definition in pure algebra, where it is assumed for rings that*

$$V \cdot U = \{\sum_{i=1}^{n} v_i \cdot u_i : n \in \mathbb{Z}^+, u_1, \ldots, u_n \in U, v_1, \ldots, v_n \in V\}.$$

Remark 2. *One can similarly define a topological right A-module with jointly continuous action and a topological two-sided A-module with jointly continuous action, taking (E,τ) to be left or two-sided A-module and requiring joint continuity of the respective module multiplications. In this paper, except for Section 8, where only the two-sided case is considered, we will give results and proofs only for the left-sided case. The results and their proofs for the right-sided case and for the two-sided case are analogous and can be carried out by the reader following the ideas of the left-sided case.*

In [2], the authors studied locally convex A-modules, i.e., the case where (A,τ_A) is a locally convex algebra and (E,τ_E) is a locally convex space. Among other things, they showed that the property of being a locally convex A-module with jointly continuous action is preserved under several algebraic and topological constructions (taking quotient modules, constructing unitization, constructing completion, taking direct product, projective limit and strict inductive limits). We generalize these results for the case of arbitrary topological algebras and arbitrary topological A-modules with jointly continuous action. In addition, we also state and prove some extra results (joint continuity of the action is preserved for submodules and direct sums; in the case of commutative topological A-bimodules, it is also preserved for topological tensor products). For continuity of the action for submodules and direct sums, see also Remarks 4.1, 4.14 and 4.15 of [2].

2. SUBMODULES AND QUOTIENT MODULES

Let (A,τ_A) be a topological algebra, (E,τ_E) a topological left A-module and (N,τ_N), where $\tau_N = \tau_E|_N$, a topological left A-submodule of (E,τ_E). This means that N is a subspace of the linear space E, $A \cdot N \subseteq N$ and $\tau_N = \{O \cap N : O \in \tau_E\}$. Let $\mathscr{N}_0$ be a base of neighbourhoods of zero in (A,τ_A) and $\mathscr{W}_0$ a base of neighbourhoods of zero in (E,τ_E). Then the collection $\mathscr{X}_0 = \{W \cap N : W \in \mathscr{W}_0\}$ is a base of neighbourhoods of zero in (N,τ_N). The converse also holds: if $\mathscr{X}_0$ is a base of neighbourhoods of zero in (N,τ_N), then there exists a base $\mathscr{W}_0$ of neighbourhoods of zero in (E,τ_E) such that $\mathscr{X}_0 = \{W \cap N : W \in \mathscr{W}_0\}$.

For each $x, y \in E$, define $x \sim y$ if and only if $x - y \in N$. Then $\sim$ is an equivalence relation on E and we can define the equivalence classes $[x] = \{y \in E : x \sim y\}$ for all $x \in E$.

Consider the quotient set $E/N = \{[x] : x \in E\}$ and the quotient map $\pi_N : E \to E/N$ which is defined by $\pi_N(x) = [x]$ for every $x \in X$. Defining $[x] + [y] = [x+y], \lambda[x] = [\lambda x]$ and $a[x] = [ax]$ for all $x, y \in E, \lambda \in \mathbb{K}$ and $a \in A$, it is easy to see that E/N is an A-module. It becomes a topological A-module if we consider on it the quotient topology $\tau_{E/N} = \{U \subset E/N : \pi_N^{-1}(U) \in \tau_E\}$.

Let $\mathscr{W}_0$ be a base of neighbourhoods of zero in (E,τ_E) and consider the collection $\mathscr{Z}_0 = \{\pi_N(W) : W \in \mathscr{W}_0\}$. Take any neighbourhood U of zero in $(E/N, \tau_{E/N})$. Then $\pi_N^{-1}(U)$ is a neighbourhood of zero in (E,τ_E). Hence, there is $W_U \in \mathscr{W}_0$ such that $W_U \subseteq \pi_N^{-1}(U)$. Notice that then $\pi_N(W_U) \in \mathscr{Z}_0$ and $\pi_N(W_U) \subseteq \pi_N(\pi_N^{-1}(U)) = U$. Hence, $\mathscr{Z}_0$ is a base of neighbourhoods of zero in $(E/N, \tau_{E/N})$.

Proposition 1. *Let (A, τ_A) be a topological algebra, (E, τ_E) a topological left A-module with jointly continuous action and (N, τ_N) a topological left A-submodule of (E, τ_E). Then the following claims hold:*
1. *(N, τ_N) is a topological left A-module with jointly continuous action;*
2. *$(E/N, \tau_{E/N})$ is a topological left A-module with jointly continuous action.*

Proof. It is obvious from the discussion preceding the proposition that (N, τ_N) and $(E/N, \tau_{E/N})$ are topological left A-modules. It only remains for us to show that the action of A on them is jointly continuous in both cases.

For that, let $\mathcal{N}_0$ be a base of neighbourhoods of zero in (A, τ_A) and $\mathcal{W}_0$ a base of neighbourhoods of zero in (E, τ_E). As the action of A on (E, τ_E) is jointly continuous, we know that for each $W \in \mathcal{W}_0$ there exist $V \in \mathcal{N}_0$ and $U \in \mathcal{W}_0$ such that $V \cdot U \subseteq W$.

(1) Consider the base $\mathcal{X}_0 = \{W \cap N : W \in \mathcal{W}_0\}$ of neighbourhoods of zero in (N, τ_N) and take an arbitrary $X \in \mathcal{X}_0$. Then there exists $W_X \in \mathcal{W}_0$ such that $X = W_X \cap N$. Now, for W_X, there exist $V_X \in \mathcal{N}_0$ and $U_X \in \mathcal{W}_0$ such that $V_X \cdot U_X \subseteq W_X$.

Set $Y_X = U_X \cap N$. Then $Y_X \in \mathcal{X}_0$ and

$$V_X \cdot Y_X = V_X \cdot (U_X \cap N) \subseteq (V_X \cdot U_X) \cap (V_X \cdot N) \subseteq W_X \cap N = X.$$

Hence, there exist $V_X \in \mathcal{N}_0$ and $Y_X \in \mathcal{X}_0$ such that $V_X \cdot Y_X \subseteq X$, which means that the action of A on (N, τ_N) is jointly continuous.

(2) Consider the base $\mathcal{Z}_0 = \{\pi_N(W) : W \in \mathcal{W}_0\}$ of neighbourhoods of zero in $(E/N, \tau_{E/N})$ and take any $Z \in \mathcal{Z}_0$. Then there exists $W_Z \in \mathcal{W}_0$ such that $Z = \pi_N(W_Z)$. Now, for W_Z, there exist $V_Z \in \mathcal{N}_0$ and $U_Z \in \mathcal{W}_0$ such that $V_Z \cdot U_Z \subseteq W_Z$.

Take $Y_Z = \pi_N(U_Z)$. Then $Y_Z \in \mathcal{Z}_0$ and

$$V_Z \cdot Y_Z = V_Z \cdot \pi_N(U_Z) = \pi_N(V_Z \cdot U_Z) \subseteq \pi_N(W_Z) = Z.$$

As $Z \in \mathcal{Z}_0$ was chosen arbitrarily, then the action of A on $(E/N, \tau_{E/N})$ is jointly continuous. □

3. UNITIZATION

One of the algebraic constructions which is used quite often is the construction of a unitization. Thus, let us start with the algebra A over $\mathbb{K}$ and consider its unitzation $A_1 = A \times \mathbb{K} = \{(a, \lambda) : a \in A, \lambda \in \mathbb{K}\}$, where the algebraic operations are defined as follows:

$$(a, \lambda) + (b, \mu) = (a + b, \lambda + \mu), \alpha(a, \lambda) = (\alpha a, \alpha\lambda), (a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$$

for all $(a, \lambda), (b, \mu) \in A_1, \alpha \in \mathbb{K}$. Under these algebraic operations, the set A_1 becomes a unital algebra over $\mathbb{K}$ with the unit element $(\theta_A, 1)$, where θ_A denotes the zero element of A. When we consider on $\mathbb{K}$ its natural topology and on A_1 the product topology $\tau_{A\times\mathbb{K}}$, we obtain a unital topological algebra $(A_1, \tau_{A\times\mathbb{K}})$ over $\mathbb{K}$ (see [1], Proposition 2.2.9, p. 87).

In the product topology $\tau_{A\times\mathbb{K}}$, we consider the base of neigbourhoods of zero in the form $\mathcal{M}_0 = \{N \times Y : N \in \mathcal{N}_0, Y \in \mathcal{K}_0\}$, where $\mathcal{N}_0$ is a base of neighbourhoods of zero in (A, τ_A), $\mathcal{K}_0 = \{B_\varepsilon : \varepsilon > 0\}$ is a base of neighbourhoods of zero in $\mathbb{K}$ and $B_\varepsilon = \{\lambda \in \mathbb{K} : \mid \lambda \mid < \varepsilon\}$.

Notice that if (E, τ_E) is a topological left A-module, then (E, τ_E) is also a topological left A_1-module, if we define the action of A_1 on (E, τ_E) by $(a, \lambda)x = ax + \lambda x$ for all $(a, \lambda) \in A_1, x \in E$.

Proposition 2. *Let (A, τ_A) be a topological algebra, $(A_1, \tau_{A\times\mathbb{K}})$ the unitization of A and (E, τ_E) a topological left A-module with jointly continuous action. Then (E, τ_E) is also a topological left A_1-module with jointly continuous action.*

Proof. As (E,τ_E) is a topological left A-module with jointly continuous action, then there exist a base $\mathscr{N}_0$ of neighbourhoods of zero in (A,τ_A) and a base $\mathscr{W}_0$ of neighbourhoods of zero in (E,τ_E) such that, for each $W \in \mathscr{W}_0$, there exist $V \in \mathscr{N}_0$ and $U \in \mathscr{W}_0$ with $V \cdot U \subseteq W$.

Take the base $\mathscr{M}_0 = \{N \times Y : N \in \mathscr{N}_0, Y \in \mathscr{K}_0\}$ of neighbourhoods of zero in A_1 and any $W \in \mathscr{W}_0$. Since the addition in (E,τ_E) is continuous and since $\mathscr{W}_0$ is a base of neighbourhoods of zero in (E,τ_E), there exists $W_1 \in \mathscr{W}_0$ such that $W_1 + W_1 \subseteq W$.

Because of the joint continuity of the action of A on (E,τ_E), there exist $V_W \in \mathscr{N}_0$ and $U_W \in \mathscr{W}_0$ such that $V_W \cdot U_W \subseteq W_1$. Since multiplication by scalars is continuous in (E,τ_E), there exist $\varepsilon_W > 0$ and $W_2 \in \mathscr{W}_0$ such that $B_{\varepsilon_W} \cdot W_2 \subseteq W_1$. Now, there is $W_3 \in \mathscr{W}_0$ such that $W_3 \subseteq U_W \cap W_2$. Notice that $M_W = V_W \times B_{\varepsilon_W} \in \mathscr{M}_0$ and

$$M_W \cdot W_3 \subseteq V_W \cdot W_3 + B_{\varepsilon_W} \cdot W_3 \subseteq V_W \cdot U_W + B_{\varepsilon_W} \cdot W_2 \subseteq W_1 + W_1 \subseteq W.$$

As $W \in \mathscr{W}_0$ was chosen arbitrarily, the action of A_1 on (E,τ_E) is jointly continuous and (E,τ_E) is a topological left A_1-module with jointly continuous action. □

4. COMPLETION

Let (E,τ_E) be a left A-module, $(I,\preceq_I),(J,\preceq_J)$ two partially ordered sets and $(x_i)_{i\in I},(y_j)_{j\in J}$ two nets that consist of elements of E. For such nets, addition, scalar multiplication and multiplication by elements of A will be defined as follows:

$$(x_i)_{i\in I} + (y_j)_{j\in J} = (x_i + y_j)_{(i,j)\in I\times J}, \;\; \lambda(x_i)_{i\in I} = (\lambda x_i)_{i\in I}, \;\; a(x_i)_{i\in I} = (ax_i)_{i\in I}$$

for all nets $(x_i)_{i\in I},(y_j)_{j\in J}$ and for all $\lambda \in \mathbb{K}, a \in A$. Two nets $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$, which consist of elements of E, are said to be equivalent, if, for any neighbourhood O of zero in (E,τ_E), there exist $i_O \in I, j_O \in J$ such that from $i_O \preceq_I i, j_O \preceq_J j$ it follows that $x_i - y_j \in O$. The fact that the nets $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ are equivalent is denoted by $(x_i)_{i\in I} \sim (y_j)_{j\in J}$. The relation $\sim$ is an equivalence relation and we will denote the class of all nets that are equivalent to $(x_i)_{i\in I}$, by $[(x_i)_{i\in I}]$. A constant net will be denoted without indices, i.e., a net $(x_i)_{i\in I}$, where $x_i = x$ for each i, will be denoted by (x) and the equivalence class of this constant net will be denoted by $[(x)]$. The completion $(\widetilde{E},\tau_{\widetilde{E}})$ of a topological A-module (E,τ_E) is the collection of all such elements which, as constant nets, are limit points of convergent nets in E. This means that an element $[(x)] \in \widetilde{E}$ if and only if there exists a partially ordered set $(I,\preceq)$ and a net $(x_i)_{i\in I} \in E$ such that the net $(x_i)_{i\in I}$ converges to x. As a topological linear space, $(\widetilde{E},\tau_{\widetilde{E}})$ is complete (see [1], Theorem 2.3.13 (iii), p. 97). If $\mathscr{W}_0$ is a base of neighbourhoods of zero in (E,τ_E), then the collection $\widetilde{\mathscr{W}_0} = \{\widetilde{W} : W \in \mathscr{W}_0\}$, where

$$\widetilde{W} = \{[(x)] \in \widetilde{E} : \text{ there exists a net } (x_i)_{i\in I} \text{ and } i_W \in I \text{ such that } (x) \sim (x_i)_{i\in I} \text{ and } x_i \in W, \text{ whenever } i_W \preceq i\}$$

is a base of neighbourhoods of zero in $(\widetilde{E},\tau_{\widetilde{E}})$. When we define the action of A on $(\widetilde{E},\tau_{\widetilde{E}})$ by taking $a[(x)] = [(ax)]$ to be the limit of the net $(ax_i)_{i\in I}$, where x is the limit of the net $(x_i)_{i\in I}$, then $(\widetilde{E},\tau_{\widetilde{E}})$ becomes a topological left A-module.

Proposition 3. *Let (A,τ_A) be a topological algebra and (E,τ_E) a topological left A-module with jointly continuous action. Then the completion $(\widetilde{E},\tau_{\widetilde{E}})$ of (E,τ_E) is also a topological left A-module with jointly continuous action.*

Proof. As in the proof of Proposition 2, there exist a base $\mathscr{N}_0$ of neighbourhoods of zero in (A,τ_A) and a base $\mathscr{W}_0$ of neighbourhoods of zero in (E,τ_E) such that for each $W \in \mathscr{W}_0$ there exist $V \in \mathscr{N}_0$ and $U \in \mathscr{W}_0$ with $V \cdot U \subseteq W$.

Consider the base $\widetilde{\mathscr{W}_0} = \{\widetilde{W} : W \in \mathscr{W}_0\}$ of neighbourhoods of zero in $(\widetilde{E},\tau_{\widetilde{E}})$ and take any $O \in \widetilde{\mathscr{W}_0}$. Then there exists $W \in \mathscr{W}_0$ such that $O = \widetilde{W}$. As addition and scalar multiplication are continuous in (E,τ_E), there exists $W_1 \in \mathscr{W}_0$ such that $-W_1 + W_1 \subseteq W$. Since the action of A is jointly continuous on (E, τ_E), there exist $V_W \in \mathscr{N}_0$ and $U_W \in \mathscr{W}_0$ such that $V_W \cdot U_W \subseteq W_1$.

Consider the set $\widetilde{U}_W \in \widetilde{\mathscr{W}_0}$ and take any $[(x)] \in \widetilde{U}_W$. Then there exists a partially ordered set $(I, \preceq)$, a net $(x_i)_{i\in I}$ and indices $i_0, i_{U_W} \in I$ such that from $i_0 \preceq i$ it follows that $x_i \in U_W$ and from $i_{U_W} \preceq i$ it follows that $x_i - x \in U_W$. Hence, $x - x_i \in -U_W$ whenever $i_0 \preceq i$. As I is partially ordered, there exists $i_1 \in I$ such that $i_0 \preceq i_1$ and $i_{U_W} \preceq i_1$. However, now $x = (x - x_{i_1}) + x_{i_1}$ and we obtain

$$V_W \cdot \{[(x)]\} = [V_W \cdot \{x\}] = [V_W \cdot \{(x - x_{i_1}) + x_{i_1}\}]$$

and

$$V_W \cdot \{(x - x_{i_1}) + x_{i_1}\} \subseteq V_W \cdot \{x - x_{i_1}\} + V_W \cdot \{x_{i_1}\}$$
$$\subseteq V_W \cdot (-U_W) + V_W \cdot U_W \subseteq -(V_W \cdot U_W) + V_W \cdot U_W \subseteq -W_1 + W_1 \subseteq W,$$

which means that $V_W \cdot \{[(x)]\} \in \widetilde{W}$. As this holds for each $[(x)] \in \widetilde{U}_W$, then $V_W \cdot \widetilde{U}_W \subseteq \widetilde{W}$ and the action of A on $(\widetilde{E}, \tau_{\widetilde{E}})$ is jointly continuous. □

5. TOPOLOGICAL DIRECT PRODUCT AND TOPOLOGICAL DIRECT SUM

Let (A, τ_A) be a topological algebra, I any set of indices and $((E_i, \tau_i))_{i\in I}$ some collection of topological left A-modules. For each $i \in I$, denote by $\mathscr{W}_i$ a base of neighbourhoods of zero in (E_i, τ_i). Then one can construct the topological direct product $(\prod_{i\in I} E_i, \tau)$ by considering the algebraic direct product $\prod_{i\in I} E_i$ of the modules $(E_i)_{i\in I}$ and equipping it with the product topology τ, in which the base of neighbourhoods of zero is the set

$$\mathscr{W}_0 = \Big\{ \prod_{i\in I} Z_i : \text{ there exist } n \in \mathbb{Z}^+ \text{ and } i_1, \ldots, i_n \in I \text{ such that } Z_i \in \mathscr{W}_i,$$
$$\text{if } i \in \{i_1, \ldots, i_n\} \text{ and } Z_i = E_i, \text{ otherwise} \Big\}. \tag{5.1}$$

It is easy to check that $(\prod_{i\in I} E_i, \tau)$ is a topological left A-module if we define the action of A on E_i by $a \cdot (x_i)_{i\in I} = (ax_i)_{i\in I}$ for all $a \in A$ and $(x_i)_{i\in I} \in \prod_{i\in I} E_i$.

Proposition 4. *Let (A, τ_A) be a topological algebra and $((E_i, \tau_i))_{i\in I}$ a collection of topological left A-modules with jointly continuous action. Then the topological direct product $(\prod_{i\in I} E_i, \tau)$ is a topological left A-module with jointly continuous action.*

Proof. As we already know that $(\prod_{i\in I} E_i, \tau)$ is a topological left A-module, it remains to show that the action of A is jointly continuous on the topological direct product.

For that, let $\mathscr{N}_0$ be a base of neighbourhoods of zero in (A, τ_A) and, for each $i \in I$, let W_i be a base of neighbourhoods of zero in (E_i, τ_i). Consider the base $\mathscr{W}_0$ of neighbourhoods of zero in $(\prod_{i\in I} E_i, \tau)$ as defined by (5.1) and take any $W = \prod_{i\in I} Z_i \in \mathscr{W}_0$. Then there exist $n \in \mathbb{Z}^+$ and $i_1, \ldots, i_n \in I$ such that

$$Z_i = \begin{cases} W_i \in \mathscr{W}_i, & \text{if } i \in \{i_1, \ldots, i_n\} \\ E_i, & \text{otherwise} \end{cases}.$$

As $(E_{i_1}, \tau_{i_1}), \ldots, (E_{i_n}, \tau_{i_n})$ are topological left A-modules with jointly continuous action, there exist $V_{i_1}, \ldots, V_{i_n} \in \mathscr{N}_0$, $U_{i_1} \in \mathscr{W}_{i_1}, \ldots, U_{i_n} \in \mathscr{W}_{i_n}$ such that $V_{i_1} \cdot U_{i_1} \subseteq W_{i_1}, \ldots, V_{i_n} \cdot U_{i_n} \subseteq W_{i_n}$. Set

$$\widetilde{V} = \bigcap_{k=i}^{n} V_{i_k} \quad \text{and } U = \prod_{i\in I} Y_i, \text{ where } Y_i = \begin{cases} U_i, & \text{if } i \in \{i_1, \ldots, i_n\} \\ E_i, & \text{otherwise} \end{cases}.$$

Then $\widetilde{V}$ is a neighbourhood of zero in (A,τ) and $U \in \mathscr{W}_0$. Hence, there exists $V \in \mathscr{N}_0$ such that $V \subseteq \widetilde{V}$.

Notice that now $V \cdot U = \prod\limits_{i\in I} V \cdot Y_i \subseteq W$ because

$$V \cdot Y_i = \begin{cases} V \cdot U_i \subseteq \widetilde{V} \cdot U_i \subseteq V_i \cdot U_i \subseteq W_i, & \text{if } i \in \{i_1, \ldots, i_n\} \\ V \cdot E_i \subseteq E_i, & \text{otherwise} \end{cases}.$$

Thus, $(\prod\limits_{i\in I} E_i, \tau)$ is a topological left A-module with jointly continuous action. □

Recall that the topological direct sum of the collection $((E_i,\tau_i))_{i\in I}$ is the pair $(\bigoplus\limits_{i\in I} E_i, \tau_{\bigoplus\limits_{i\in I} E_i})$, where $\bigoplus\limits_{i\in I} E_i$ consists of all such elements $(x_i)_{i\in I} \in \prod\limits_{i\in I} E_i$, where $x_i \in E_i$ for all $i \in I$ and $x_i \neq \theta_{E_i}$ only for finitely many values of $i \in I$. The topology $\tau_{\bigoplus\limits_{i\in I} E_i} = \tau|_{\bigoplus\limits_{i\in I} E_i}$ is the restriction of the direct product topology on the direct sum.

Remark 3. *In general, there are several different topologies that one could consider on an algebraic direct sum – for example, the box topology, the asterisk topology, etc. In this paper, we consider only the case where the topology on the direct sum is the subspace topology of the direct product topology, because it makes the main result about the topological direct sum an easy corollary of the preceding results. In the case of other topologies, more work should be done in order to obtain the result.*

Since the algebraic direct sum of left A-modules $(E_i)_{i\in I}$ is a subset of the algebraic direct product of the same left A-modules, we define the action of A on the algebraic direct sum as we did it for any element of the algebraic direct product. It is easy to check that then the algebraic direct sum of left A-modules becomes itself a left A-module, which is closed with respect to the algebraic operations. Moreover, as we are considering the subspace topology, it will become a topological left A-module, hence, a topological left A-submodule of the topological direct product of the same collection of topological left A-modules.

Now, as a consequence of Proposition 1 and Proposition 4, we obtain the following result, where $\tau_{\bigoplus\limits_{i\in i} E_i}$ is the restriction of the direct product topology on the direct sum.

Corollary 1. *Let (A,τ_A) be a topological algebra, $((E_i,\tau_i))_{i\in I}$ a collection of topological left A-modules with jointly continuous action and $(\bigoplus\limits_{i\in I} E_i, \tau_{\bigoplus\limits_{i\in I} E_i})$ the topological direct sum of the collection $((E_i,\tau_i))_{i\in I}$ of topological left A-modules. Then $(\bigoplus\limits_{i\in I} E_i, \tau_{\bigoplus\limits_{i\in I} E_i})$ is also a topological left A-module with jointly continuous action.*

Proof. As we already mentioned before stating the corollary, the direct sum of the collection $((E_i,\tau_i))_{i\in I}$ of topological left A-modules is a topological left A-submodule of the direct product $(\prod\limits_{i\in I} E_i, \tau)$ of the same collection of topological left A-modules.

By Proposition 4, we know that the action of A on the direct product $(\prod\limits_{i\in I} E_i, \tau)$ is jointly continuous. By Proposition 1, we obtain now that the action of A on the direct sum $(\bigoplus\limits_{i\in I} E_i, \tau_{\bigoplus\limits_{i\in I} E_i})$ is also jointly continuous. □

6. PROJECTIVE LIMIT

Similarly to the case of direct product, one can consider also a topological projective limit of a family of topological left A-modules. In this case we need not just a set I but a partially ordered set $(I,\preceq)$, a collection $((E_i,\tau_i))_{i\in I}$ of topological left A-modules and a collection $\{h_{ij} : E_j \to E_i, i, j \in I, i \preceq j\}$ of continuous A-linear maps such that $h_{ii} : E_i \to E_i$ is the identity map for each $i \in I$ and $h_{ik} = h_{ij} \circ h_{jk}$ for each $i, j, k \in I$

with $i \preceq j \preceq k$. Such a collection $\{((E_i,\tau_i))_{i\in I};(h_{ij})_{i,j\in I,i\preceq j}\}$ is called a projective system of topological left A-modules.

The topological projective limit of the projective system $\{((E_i,\tau_i))_{i\in I};(h_{ij})_{i,j\in I,i\preceq j}\}$ of topological left A-modules is the topological space $(\varprojlim E_i, \tau_{\varprojlim E_i})$, where

$$\varprojlim E_i = \{(x_i)_{i\in I} \in \prod_{i\in I} E_i : h_{ij}(x_j) = x_i \text{ for all } i,j\in I \text{ with } i\preceq j\}$$

and the topology $\tau_{\varprojlim E_i} = \tau|_{\varprojlim E_i}$ is the restriction of the product topology τ on $\prod_{i\in I} E_i$ to the algebraic projective limit set $\varprojlim E_i$.

Consider the projection maps $p_k : \prod_{i\in I} E_i \to E_k$, defined by $p_k((x_i)_{i\in I}) = x_k$ for each $(x_i)_{i\in I} \in \prod_{i\in I} E_i$ and each $k\in I$. It is known that the projection maps $(p_k)_{k\in I}$ are continuous with respect to the product topology. It is also easy to see that $p_i = h_{ij}\circ p_j$ for all $i,j\in I$ with $i\preceq j$.

In the case of the topological projective limit, the restrictions $\pi_k = p_k|_{\varprojlim E_i}$ of the projection maps are considered for each $k\in I$. Since $p_i = h_{ij}\circ p_j$, then also $\pi_i = p_i|_{\varprojlim E_i} = (h_{ij}\circ p_j)|_{\varprojlim E_i} = h_{ij}\circ(p_j|_{\varprojlim E_i}) = h_{ij}\circ\pi_j$ for all $i,j\in I$ with $i\preceq j$.

As the direct product of topological left A-modules is a left A-module, then the topological projective limit, as a subset, is also a topological left A-module, because the topology on the topological projective limit is the subspace topology and the algebraic operations on the direct product carry over to the projective limit easily.

Denote a base of neighbourhoods of zero in (E_i,τ_i) by $\mathscr{W}_i$ for each $i\in I$. Then, it is a well-known fact that a base of neighbourhoods of zero in $(\varprojlim E_i, \tau_{\varprojlim E_i})$ has the form

$$\mathscr{W}_{\varprojlim E_i} = \left\{ \bigcap_{k=1}^{n} \pi_{i_k}^{-1}(W_{i_k}) : n\in\mathbb{Z}^+, i_1,\dots,i_n\in I, W_{i_1}\in\mathscr{W}_{i_1},\dots,W_{i_n}\in\mathscr{W}_{i_n} \right\}.$$

Proposition 5. *Let (A,τ) be a topological algebra and $\{((E_i,\tau_i))_{i\in I};(h_{ij})_{i,j\in I,i\preceq j}\}$ a projective system of topological left A-modules with jointly continuous action. Then the topological projective limit $(\varprojlim E_i, \tau_{\varprojlim E_i})$ is also a topological left A-module with jointly continuous action.*

Proof. Since we already know that $(\varprojlim E_i, \tau_{\varprojlim E_i})$ is a topological left A-module, it only remains for us to prove that the action of A is jointly continuous on the topological projective limit.

For that, consider the base $\mathscr{W}_{\varprojlim E_i}$ of neighbourhoods of zero in $(\varprojlim E_i, \tau_{\varprojlim E_i})$ and take any $W\in\mathscr{W}_{\varprojlim E_i}$. Then there exist $n\in\mathbb{Z}^+, i_1,\dots,i_n\in I$ and $W_{i_1}\in\mathscr{W}_{i_1},\dots,W_{i_n}\in\mathscr{W}_{i_n}$ such that $W = \bigcap_{k=1}^{n}\pi_{i_k}^{-1}(W_{i_k})$. Since the action of A is jointly continuous on the A-modules $(E_{i_1},\tau_{i_1}),\dots,(E_{i_n},\tau_{i_n})$, there exist $V_{i_1},\dots,V_{i_n}\in\mathscr{N}_0$, $U_{i_1}\in\mathscr{W}_{i_1},\dots,U_{i_n}\in\mathscr{W}_{i_n}$ such that $V_{i_1}\cdot U_{i_1}\subseteq W_{i_1},\dots,V_{i_n}\cdot U_{i_n}\in W_{i_n}$.

Take $\widetilde{V} = \bigcap_{k=1}^{n} V_{i_k}$ and $U = \bigcap_{k=1}^{n}\pi_{i_k}^{-1}(U_{i_k})$. Then $\widetilde{V}$ is a neighbourhood of zero in (A,τ_A) and $U\in\mathscr{W}_{\varprojlim E_i}$. Hence, there exists $V\in\mathscr{N}_0$ such that $V\subseteq\widetilde{V}$. Now,

$$V\cdot U \subseteq \bigcap_{k=1}^{n}\widetilde{V}\cdot\pi_{i_k}^{-1}(U_{i_k}) \subseteq \bigcap_{k=1}^{n} V_{i_k}\cdot\pi_{i_k}^{-1}(U_{i_k}) \subseteq \bigcap_{k=1}^{n}\pi_{i_k}^{-1}(V_{i_k}\cdot U_{i_k}) \subseteq \bigcap_{k=1}^{n}\pi_{i_k}^{-1}(W_{i_k}) = W.$$

As $W\in\mathscr{W}_{\varprojlim E_i}$ was chosen arbitrarily, the action of A on $(\varprojlim E_i, \tau_{\varprojlim E_i})$ is jointly continuous. □

7. INDUCTIVE LIMIT

Let $(I,\preceq)$ be a partially ordered set and $((E_i,\tau_i))_{i\in I}$ a collection of topological left A-modules. Suppose that there exists a collection $\{h_{ij}: E_i\to E_j, i,j\in I, i\preceq j\}$ of continuous A-linear maps such that $h_{ii}: E_i\to E_i$ is the identity map for each $i\in I$ and $h_{ik}=h_{jk}\circ h_{ij}$ for each $i,j,k\in I$ with $i\preceq j\preceq k$. Such a collection $\{((E_i,\tau_i))_{i\in I};(h_{ij})_{i,j\in I,i\preceq j}\}$ is called an inductive system of topological left A-modules.

In the set $\bigcup\limits_{i\in I}E_i$ an equivalence relation $\sim$ is defined, where for $x_1,x_2\in\bigcup\limits_{i\in I}E_i$ and $i_1,i_2\in I$, with $x_1\in E_{i_1}$, $x_2\in E_{i_2}$, we have $x_1\sim x_2$ if and only if there exists $i\in I$ such that $i_1\preceq i, i_2\preceq i$ and $h_{i_1 i}(x_1)=h_{i_2 i}(x_2)$. Denote by $[x]$ the equivalence class of $x\in\bigcup\limits_{i\in I}E_i$ under the relation $\sim$. The set $\underrightarrow{\lim}E_i=(\bigcup\limits_{i\in I}E_i)/\sim$ of all equivalence classes is called the inductive limit of sets $(E_i)_{i\in I}$. Let $\pi:\bigcup\limits_{i\in I}E_i\to\underrightarrow{\lim}E_i$ be the natural quotient map, defined by $\pi(x)=[x]$ for each $x\in\bigcup\limits_{i\in I}E_i$ and for each $k\in I$, let $\nu_k: E_k\to\bigcup\limits_{i\in I}E_i$ be the inclusion map. Then, for each $k\in I$, the map $h_k=\pi\circ\nu_k: E_k\to\underrightarrow{\lim}E_i$ is such a map that $\underrightarrow{\lim}E_i=\bigcup\limits_{i\in I}h_i(E_i)$.

Remark 4. *Some authors construct the inductive limit of a direct system of modules as a quotient module of the direct sum of the family, as one of the referees kindly noted. The approach presented in this paper for constructing the inductive limit using equivalence relation is chosen to make this construction more independent of the construction of the direct sum and to give a different proof for Proposition 6.*

The topological inductive limit of the inductive system $\{(E_i,\tau_i)_{i\in I};(h_{ij})_{i,j\in I,i\preceq j}\}$ of topological left A-modules is the topological space $(\underrightarrow{\lim}E_i,\tau_{\underrightarrow{\lim}E_i})$, where the topology

$$\tau_{\underrightarrow{\lim}E_i}=\{U\subseteq\underrightarrow{\lim}E_i: h_i^{-1}(U)\in\tau_i\ \text{ for each } i\in I\}$$

is the final topology, defined by the maps $(h_i)_{i\in I}$. With respect to this topology, all maps $(h_i)_{i\in I}$ and also the quotient map π are continuous.

Notice that algebraic operations on the inductive limit are defined by “lifting” the elements into the suitable space E_i, where the operations are defined. Indeed, for $a\in A,\lambda\in\mathbb{K}$ and $[x_1],[x_2]\in\underrightarrow{\lim}E_i$, there exist $i_1,i_2,j\in I$ such that $x_1\in E_{i_1},x_2\in E_{i_2},i_1\preceq j,i_2\preceq j$. Hence, there exist $y_1=h_{i_1 j}(x_1)=h_{jj}(y_1)$, $y_2=h_{i_2 j}(x_2)=h_{jj}(y_2)\in E_j$ such that $[x_1]=[y_1]$ and $[x_2]=[y_2]$.

Take any $k\in I$ such that $i_1\preceq k,i_2\preceq k$ and $z_1=h_{i_1 k}(x_1),z_2=h_{i_2 k}(x_2)$. Then there exists $l\in I$ such that $j\preceq l,k\preceq l$ and we have

$$h_{jl}(y_1)=(h_{jl}\circ h_{i_1 j})(x_1)=h_{i_1 l}(x_1)=(h_{kl}\circ h_{i_1 k})(x_1)=h_{kl}(z_1)$$

and similarly $h_{jl}(y_2)=h_{kl}(z_2)$. Hence, $y_1\sim z_1$ and $y_2\sim z_2$, which means that $[y_1]=[z_1]$ and $[y_2]=[z_2]$.

Therefore, we can define $[x_1]+[x_2]=[y_1+y_2],\lambda[x_1]=[\lambda x_1]=[\lambda y_1]$ and $a[x_1]=[ax_1]=[ay_1]$. With respect to these algebraic operations, the topological inductive limit becomes a left A-module. Moreover, due to the definition of the topology on the inductive limit, it also becomes a topological left A-module.

Denote module multiplication on (E_i,τ_i) by m_i for each $i\in I$ and module multiplication on $\underrightarrow{\lim}E_i$ by m, i.e., $m_i: A\times E_i\to E_i$ is defined by $m_i(a,x_i)=ax_i$ for all $(a,x_i)\in A\times E_i$ and $m: A\times\underrightarrow{\lim}E_i\to\underrightarrow{\lim}E_i$ is defined by

$$m(a,[x])=[m_i(a,x_i)]=[ax_i]$$

for any $i\in I$ such that $x_i\in E_i$ and $[x]=[x_i]$. Suppose that there exist $i,j\in I$ such that $i\neq j$ and $[x_i]=[x]=[x_j]$. Then there is $k\in I$ such that $i\preceq k,j\preceq k$ and $h_{ik}(x_i)=h_{jk}(x_j)$. Now, $h_{ik}(ax_i)=ah_{ik}(x_i)=ah_{jk}(x_j)=h_{jk}(ax_j)$ and $[m_i(a,x_i)]=[ax_i]=[ax_j]=[m_j(a,x_j)]$. Hence, the result of module multiplication does not depend on the selection of $i\in I$ with $[x_i]=[x]$.

In [2], Proposition 6 of the present paper was not proved nor even stated (Proposition 4.16 of [2], which is similar to Proposition 6 of the present paper, was stated only for the strict inductive limit). Here we offer a proof of that result, but with different methods than the previous ones, which used a base of neighbourhoods of zero. In order to prove the result, we first need the following lemma.

Lemma 1. *Let (A,τ) be a topological algebra, $(\varinjlim E_i, \tau_{\varinjlim E_i})$ the topological inductive limit of the inductive system $\{((E_i,\tau_i))_{i\in I};(h_{ij})_{i,j\in I,i\preceq j}\}$ of topological left A-modules and $m: A\times \varinjlim E_i \to \varinjlim E_i$ a bilinear map. If the map $m\circ(\mathrm{id}_A\times h_i): A\times E_i\to E_i$, where id_A is the identity map on A, is continuous for each $i\in I$, then the map m is also continuous.*

Proof. Since we are dealing with A-modules and m is a bilinear map, then it suffices to check the continuity of the maps at the zero element (see [3], p. 171) of $A\times\varinjlim E_i$. Thus, suppose that $m\circ(\mathrm{id}_A\times h_i)$ is continuous at (θ_A,θ_{E_i}) for each $i\in I$. Let O be any neighbourhood of zero in $(\varinjlim E_i,\tau_{\varinjlim E_i})$. Then, for each $i\in I$, there exist a neighbourhood V_i of θ_A and a neighbourhood U_i of θ_{E_i} such that $(m\circ(\mathrm{id}_A\times h_i))(V_i\times U_i)\subseteq O$.

Take $P=\bigcup\limits_{i\in I}(V_i\times h_i(U_i))$ and let $p_A: A\times\varinjlim E_i\to A, p_{\varinjlim E_i}: A\times\varinjlim E_i\to\varinjlim E_i$ be the projections. Then $p_A(P)=\bigcup\limits_{i\in I}V_i$ is a neighbourhood of θ_A, as a union of neighbourhoods of θ_A, and $p_{\varinjlim E_i}(P)=\bigcup\limits_{i\in I}h_i(U_i)$ is a neighbourhood of $\theta_{\varinjlim E_i}$, because $h_k^{-1}(\bigcup\limits_{i\in I}h_i(U_i))\supseteq U_k$ is a neighbourhood of θ_{E_k} for each $k\in I$. Hence, P is a neighbourhood of $(\theta_A,\theta_{\varinjlim E_i})$ in the product topology on $A\times\varinjlim E_i$. This means that there exists a neighbourhood V of zero in the base of neighbourhoods of θ_A in τ_A and a neighbourhood U of zero in the base of neighbourhoods of $\theta_{\varinjlim E_i}$ in $\tau_{\varinjlim E_i}$ such that $V\times U\subseteq P$.

Take any $(a,[x])\in V\times U$. Then $(a,[x])\in P$, which means that there exists $k\in I$ such that $(a,[x])\in V_k\times h_k(U_k)$. Hence, there exists $x_k\in U_k$ such that $[x]=h_k(x_k)$, $(a,x_k)\in V_k\times U_k$ and

$$m(a,[x])=m(a,h_k(x_k))=m((\mathrm{id}_A\times h_k)(a,x_k))=(m\circ(\mathrm{id}_A\times h_k))(a,x_k)\subseteq(m\circ(\mathrm{id}_A\times h_k))(V_k\times U_k)\subseteq O.$$

Hence, $m(V\times U)\subseteq O$ and the map m is jointly continuous. □

Using this lemma, we obtain the desired result for the topological inductive limit.

Proposition 6. *Let (A,τ_A) be a topological algebra and $\{(E_i,\tau_i)_{i\in I};(h_{ij})_{i,j\in I,i\preceq j}\}$ an inductive system of topological left A-modules with jointly continuous action. Then the topological inductive limit $(\varinjlim E_i,\tau_{\varinjlim E_i})$ is also a topological left A-module with jointly continuous action.*

Proof. Notice that

$$(m\circ(\mathrm{id}_A\times h_i))(a,x_i)=m(a,h_i(x_i))=m(a,[x_i])=[m_i(a,x_i)]=\pi(m_i(a,x_i))=(\pi\circ m_i)(a,x_i)$$

for each $i\in I, a\in A$ and $x_i\in E_i$. Since the quotient map π and the map m_i are continuous, then $m\circ(\mathrm{id}_A\times h_i)$ is also continuous for each $i\in I$. Using Lemma 1, we obtain that the map m is also continuous. Hence, the action of A on the topological inductive limit is jointly continuous. □

8. TOPOLOGICAL TENSOR PRODUCT

Let A be an algebra and $(M,\cdot_l,\cdot_r)$ an A-bimodule, i.e., $(M,\cdot_l)$ is a left A-module, $(M,\cdot_r)$ is a right A-module and $(a\cdot_l m)\cdot_r b=a\cdot_l(m\cdot_r b)$ for all $a,b\in A, m\in M$. Then we know that, for any $a\in A$ and $m\in M$, we can calculate $a\cdot_l m\in M$ and $m\cdot_r a\in M$, but it might occur that $a\cdot_l m\neq m\cdot_r a$. In what follows, we also need that condition to be true.

Definition 1. *An A-bimodule $(M,\cdot_l,\cdot_r)$ is called a* **commutative** *A*-**bimodule** *if $a\cdot_l m=m\cdot_r a$ for each $a\in A$ and each $m\in M$. In case (A,τ_A) is a topological algebra and $(M,\cdot_l,\cdot_r)$ is also a topological A-bimodule, we call M a* **commutative topological** *A*-**bimodule**.

In what follows, we will not use the subindices l nad r for module multiplication. This will make the text more concise and, hopefully, will not cause any confusion.

Let I be any nonempty set of indices and $((E_i,\tau_i))_{i\in I}$ a collection of commutative topological A-bimodules. Then the topological tensor product of the collection $((E_i,\tau_i))_{i\in I}$ is the pair $(\bigotimes\limits_{i\in I}E_i,\tau_{\bigotimes\limits_{i\in I}E_i})$, where $\bigotimes\limits_{i\in I}E_i$ is the algebraic tensor product of A-bimodules $(E_i)_{i\in I}$ and $\tau_{\bigotimes\limits_{i\in I}E_i}$ is the tensor product topology, a base of which consists of sets in the form $\bigotimes\limits_{i\in I}O_i$, where, for each $i\in I$, O_i is from the base of the topology τ_i.

Notice that (by the definition of an element of a tensor product of algebras or even linear spaces) an arbitrary element $x\in\bigotimes\limits_{i\in I}E_i$ can be represented in the form $\sum\limits_{k=1}^{n_x}\bigotimes\limits_{i\in I}x_{i,k}$ of a finite sum of elementary tensors $\bigotimes\limits_{i\in I}x_{i,k}$, where $n_x\in\mathbb{Z}^+$ and $x_{i,k}\in E_i$ for each $i\in I$ and $k\in\{1,\dots,n_x\}$.

Fix any $i_0\in I$ and, for $a,b\in A$, define $a\cdot x\cdot b=\sum\limits_{k=1}^{n_x}\bigotimes\limits_{i\in I}y_{i,k}$, where $y_{i_0,k}=ax_{i_0,k}b$ and $y_{i,k}=x_{i,k}$ for each $i\in I\setminus\{i_0\}$. The result of the multiplication $a\cdot xb$ does not depend on the choice of $i_0\in I$, because for any $x=\sum\limits_{k=1}^{n_x}\bigotimes\limits_{i\in I}x_{i,k}$, $i_1\in I$ and $a,b\in A$, we have for each $k\in\{1,\dots,n_x\}$ the equalities

$$\cdots\otimes ax_{i_0,k}b\otimes\cdots\otimes x_{i_1,k}\otimes\cdots=\cdots\otimes x_{i_0,k}ba\otimes\cdots\otimes x_{i_1,k}\otimes\cdots=\dots$$

$$=\cdots\otimes x_{i_0,k}\otimes\cdots\otimes bax_{i_1,k}\otimes\cdots=\cdots\otimes x_{i_0,k}\otimes\cdots\otimes ax_{i_1,k}b\otimes\dots,$$

by using the property of the tensor product of A-modules sufficiently many times, which allows us to “flip” or “toss” any element of A from one side of the tensor product sign to another. (We can write $x_{i_1,k}$ to the right-hand side of $x_{i_0,k}$ in the tensor product because the set I is not ordered and we can freely choose the order in which we write the factors in the tensor product.)

By the same property of “flipping” or “tossing” the elements of the topological algebra (A,τ) from one side of the tensor product sign to another, we can show that the topological tensor product of the collection of commutative topological A-bimodules is itsef also a commutative topological A-bimodule. With that, we are ready for the last result of this paper.

Proposition 7. *Let (A,τ_A) be a topological algebra, I any nonempty set of indices, and $((E_i,\tau_i))_{i\in I}$ a collection of commutative topological A-bimodules. If there exists $i_0\in I$ such that the two-sided action of A on (E_{i_0},τ_{i_0}) is jointly continuous, then the topological tensor product $(\bigotimes\limits_{i\in I}E_i,\tau_{\bigotimes\limits_{i\in I}E_i})$ is also a commutative topological A-bimodule with jointly continuous two-sided action.*

Proof. We already know that the topological tensor product $(\bigotimes\limits_{i\in I}E_i,\tau_{\bigotimes\limits_{i\in I}E_i})$ of the collection $((E_i,\tau_i))_{i\in I}$ of commutative topological A-bimodules is a commutative topological A-bimodule. Let us show that the two-sided action of A on the topological tensor product is also jointly continuous.

Let $\mathscr{N}_0$ be a base of neighbourhoods of zero in (A,τ_A), $\mathscr{L}_i$ be a base of neighbourhoods of zero in (E_i,τ_i), for each $i\in I$, and consider the base

$$\mathscr{L}_0=\Big\{\bigotimes_{i\in I}O_i : O_i\in\mathscr{L}_i \text{ for each } i\in I\Big\}$$

of neighbourhoods of zero in $(\bigotimes\limits_{i\in I}E_i,\tau_{\bigotimes\limits_{i\in I}E_i})$.

Let $i_0\in I$ be such an index that the two-sided action of A on (E_{i_0},τ_{i_0}) is jointly continuous. Take any $O\in\mathscr{L}_0$. Then there exist neighbourhoods of zero $O_i\in\mathscr{L}_i, i\in I$ such that $O=\bigotimes\limits_{i\in I}O_i$. Note that an element y belongs to O if and only if there exist $n_y\in\mathbb{Z}^+$ and, for each $i\in I$, elements $y_{i,1},\dots,y_{i,n_y}\in O_i$ such that $y=\sum\limits_{k=1}^{n_y}\bigotimes\limits_{i\in I}y_{i,k}$.

Since the two-sided action of A on (E_{i_0}, τ_{i_0}) is jointly continuous, then there exist $V_{i_0}, T_{i_0} \in \mathscr{N}_0$ and $U_{i_0} \in \mathscr{Z}_{i_0}$ such that $V_{i_0} \cdot U_{i_0} \cdot T_{i_0} \subseteq O_{i_0}$. Set $V = V_{i_0}, T = T_{i_0}$ and $U = \bigotimes_{i \in I} W_i$, where $W_{i_0} = U_{i_0}$ and $W_i = O_i$ for each $i \in I \setminus \{i_0\}$. Then $V, T \in \mathscr{N}_0$ and $U \in \mathscr{Z}_0$. Take any $a \in V, b \in T$ and $x \in U$. Then there exist $n_x \in \mathbb{Z}^+$ and, for each $i \in I$, elements $x_{i,1}, \ldots, x_{i,n_x} \in W_i$ such that $x = \sum_{k=1}^{n_x} \bigotimes_{i \in I} x_{i,k}$. Notice that $a \in V_{i_0}, b \in T_{i_0}$ and $x_{i_0,k} \in U_{i_0}$ for each $k \in \{1, \ldots, n_x\}$. Thus, $ax_{i_0,k}b \in V_{i_0} \cdot U_{i_0} \cdot T_{i_0} \subseteq O_{i_0}$ for each $k \in \{1, \ldots, n_x\}$. For each $i \in I \setminus \{i_0\}$, we have $x_{i,k} \in O_i$, for any $k \in \{1, \ldots, n_x\}$. For each $k \in \{1, \ldots, n_x\}$, set $y_{i_0,k} = ax_{i_0,k}b$ and $y_{i,k} = x_{i,k}$, if $i \in I \setminus \{i_0\}$. Take $y = a \cdot x \cdot b$ and $n_y = n_x$. Then

$$a \cdot x \cdot b = y = \sum_{k=1}^{n_y} \bigotimes_{i \in I} y_{i,k},$$

with $y_{i,k} \in O_i$ for each $i \in I$ and $k \in \{1, \ldots, n_y\}$.

Hence, $a \cdot x \cdot b \in O$. As $a \in V, b \in T$ and $x \in U$ were chosen arbitrarily, we see that $V \cdot U \cdot T \subseteq O$ and the two-sided action of A on the tensor product is jointly continuous. □

Open problem. Is there any possibility of obtaining a result similar to Proposition 7 for the collection $((E_i, \tau_i))_{i \in I}$ of one-sided A-modules?

9. CONCLUSION

In the present paper we have shown that the property of topological A-modules to have jointly continuous module action is inherited under several algebraic and topological constructions.

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Moodulkorrutamise ühtsest pidevusest

Mart Abel

Artiklis näidatakse, et mitmed algebralised ja topoloogilised konstruktsioonid (faktormooduli võtmine, otsekorrutise, otsesumma, projektiivse piiri, injektiivse piiri või tensorkorrutise moodustamine jne) säilitavad topoloogiliste A-moodulite moodulkorrutamise ühtse pidevuse.