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MECHANICS

Solitary waves, shock waves and conservation laws with the surface tension effect in the Boussinesq equation

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Abstract. This paper secures solitary waves, shock waves and singular solitary waves for the Boussinesq equation, which is studied with the inclusion of surface tension. The method of undetermined coefficients has yielded such waves. The Lie symmetry analysis has introduced a fresh perspective to the model. Conserved densities and corresponding conserved quantities are computed using the multiplier approach.

Keywords: solitary waves, surface tension, Lie symmetry, conservation laws.

1. INTRODUCTION

There are several models that govern the dynamics of shallow water waves along lake shores and seashores. The most noticeable is the Korteweg–de Vries (KdV) equation. While innumerable models have been addressed in this context, none of them has taken into consideration the effect of surface tension. One of the models that also describes the flow of shallow water waves is the Boussinesq equation (BE). It is only this BE that was

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revisited by Daripa a couple of decades ago to include the effect of surface tension, and thus the sixth-order BE (6BE) was derived from the first principles [3]. The existence and uniqueness of solutions to the 6BE has been established [4,5,7,9,10]. Conservation laws for this model have been studied without conserved quantities [8].

The current paper will address several aspects of the 6BE. This model will be studied with a generalized flavor to derive solutions of solitary waves, shock waves and singular solitary waves for the model. For derivation of these wave solutions, the method of undetermined coefficients would be implemented. Subsequently, the Lie symmetry analysis would lead to the reduced ordinary differential equation (ODE), which will result in the failure of its further analysis because of its complex structure. Finally, the multiplier approach would lead to the retrieval of conserved quantities, for the general power law as well as for the special case where the power law parameter is condensed to unity. These would finally compute the conserved quantity based on the derived solitary wave solution. The detailed analyses will be presented in the remainder of the paper after a succinct revisitation to the model.

1.1. Governing model

The 6BE in its dimensionless form reads as

$$q_{tt} - k^2 q_{xx} + c\left(q^{2n}\right)_{xx} + a_1 q_{xxxx} + a_2 q_{xxtt} + b_1 q_{xxxxxx} + b_2 q_{xxxxtt} = 0. \tag{1}$$

Here, the independent variables are x and t, which represent spatial and temporal variables, respectively. Then the dependent variable $q(x,t)$ denotes the wave form. The first two terms represent the wave operator, with k being the wave number. The nonlinearity coefficient is designated by c, with the parameter n being the general power law parameter, and this parameter gives a generalized flavor to the model. The special case with $n = 1$ has been studied on several ocassions with the BE [3–10]. Next, the coefficients a_j and b_j for $j = 1,2$ indicate the coefficients of the fourth- and sixth-order dispersion terms, respectively. The coefficients of b_j stem from the surface tension effect. The fourth-order dispersion terms due to the effects of a_j are derived in the original model from the Navier–Stokes equation, following the first principles. This current model (1) will be analyzed in detail along with the derivation of solitary waves, shock waves and conservation laws. The details will be provided in the subsequent sections.

2. WAVE SOLUTIONS

This section implements the method of undetermined coefficients to derive solitary waves, shock waves and singular solitary waves. The parametric constraints naturally emerge from the solution structure of such waves. These are also presented and enumerated. The four subsections focus on the derivation of these waves, supported by the 6BE.

2.1. Solitary waves

This subsection deals with solitary wave solutions to the BE, with the presence of surface tension (1). To proceed, we first assume the solution in the form

$$q(x,t) = A\,\mathrm{sech}^p[B(x - vt)], \tag{2}$$

where A describes the wave amplitude, B represents the inverse width and v is the corresponding wave velocity, while the parameter p will be determined by proper balance. Thus, substituting the ansatz (2) into the considered system (1) yields

$$-p^2[k^2-v^2-p^2(a_1+a_2v^2+p^2(b_1+b_2v^2)B^2)B^2]AB^2\,\mathrm{sech}^p\,\tau+p(p+1)\Big\{k^2-v^2-(2(2+p(2+p))$$
$$\times[a_1+a_2v^2]+(4+p(p+2))(4+3p(p+2))[b_1+b_2v^2]B^2)B^2\Big\}AB^2\,\mathrm{sech}^{p+2}\,\tau+p(p+1)(p+2)(p+3)$$
$$\times[a_1+a_2v^2+(20+3p(p+4))(b_1+b_2v^2)B^2]AB^4\,\mathrm{sech}^{p+4}\,\tau-p(p+1)(p+2)(p+3)(p+4)(p+5)$$
$$\times(b_1+b_2v^2)AB^6\,\mathrm{sech}^{p+6}\,\tau-2cnp(2np+1)A^{2n}B^2\,\mathrm{sech}^{2np+2}\,\tau+4cn^2p^2A^{2n}B^2\,\mathrm{sech}^{2np}\,\tau=0, \tag{3}$$

where the notation $\tau=B(x-vt)$ has been adopted. In order to determine the value of the parameter p, we balance $2np+2$ with $p+4$, or $2np$ with $p+2$, giving in both cases

$$p=\frac{2}{2n-1}, \tag{4}$$

where trivially $n\neq 1/2$. Then, setting the coefficients of the linearly independent functions $\mathrm{sech}^{\frac{j}{2n-1}}\,\tau$ to zero for $j=2,4n,8n-2,$ and $12n-4,$ one obtains for the amplitude

$$A=\left[\frac{(2n+1)(b_1+b_2k^2)}{2cb_2}\right]^{\frac{1}{2n-1}} \tag{5}$$

as long as

$$cb_2\left(b_1+b_2k^2\right)>0 \tag{6}$$

is satisfied, and for the wave velocity

$$v=\sqrt{-\frac{b_1}{b_2}}, \tag{7}$$

subject to the constraint

$$b_1b_2<0. \tag{8}$$

This shows that solitary waves for the 6BE will exist provided that the two sixth-order dispersion terms bear opposite signs.

Similarly, in view of (7), the inverse width proves to be

$$B=\frac{|2n-1|}{2}\sqrt{\frac{b_1+b_2k^2}{a_1b_2-a_2b_1}} \tag{9}$$

provided that

$$\left(b_1+b_2k^2\right)(a_1b_2-a_2b_1)>0 \tag{10}$$

remains valid. Another immediate observation yields

$$n>\frac{1}{2}. \tag{11}$$

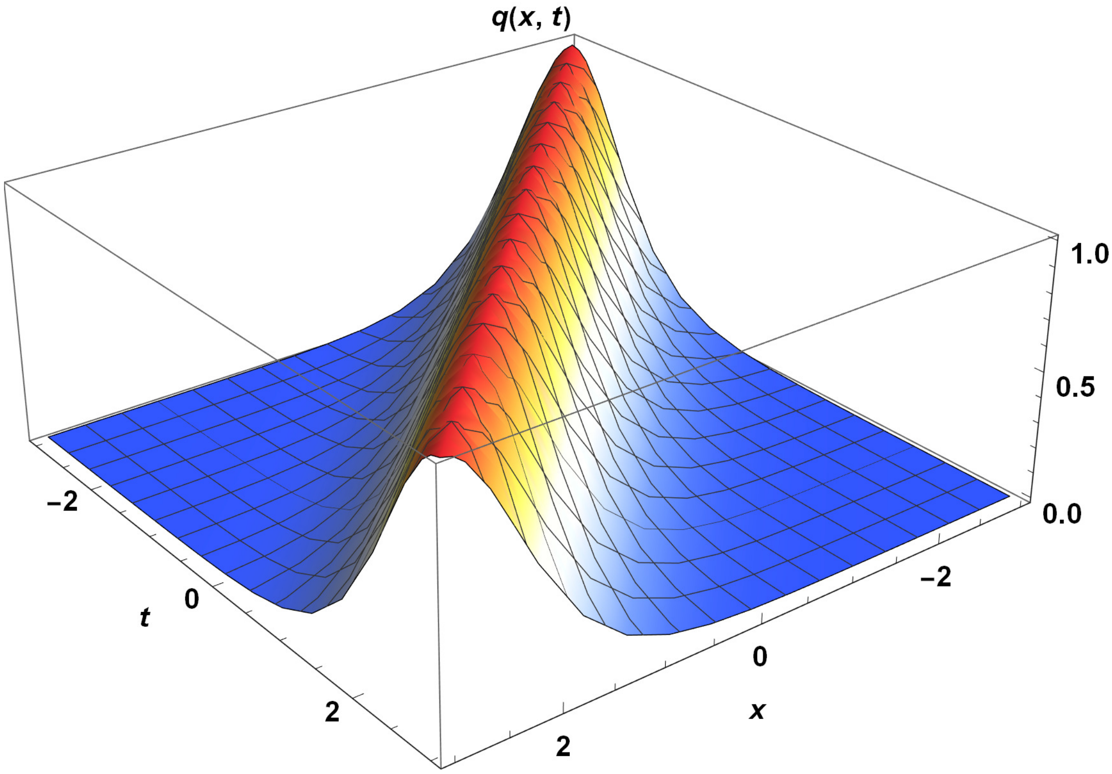

Fig. 1. Profile of a solitary wave (12) with $n=1$, $b_1=1$, $b_2=-2$, $k=1$, $c=1$, $a_1=1$ and $a_2=1$.

Thus, nonlinearity of the power law implicates the existence of solitary waves as long as the restriction on n remains valid.

Therefore, the solitary wave solution to the 6BE, with surface tension, (1) is given by

$$q(x,t) = A\,\mathrm{sech}^{\frac{2}{2n-1}}[B(x-vt)], \tag{12}$$

where the corresponding amplitude, inverse width and velocity are given in (5), (9) and (7), respectively. The solution will exist whenever all the conditions (6), (10) and (8) are satisfied. Figure 1 depicts the surface plot of a solitary wave.

2.2. Shock waves

In this section we explore shock wave solutions supported by the system (1). The initial hypothesis to be considered in this situation is

$$q(x,t) = A\tanh^{p}[B(x-vt)], \tag{13}$$

where, as for solitary waves, A represents the amplitude, B the inverse width, v is the wave velocity, and p a parameter to be determined. The substitution of (13) into (1) yields in simplified form

$$p(p-1)(p-2)(p-3)(p-4)(p-5)(b_1+b_2v^2)AB^6\tanh^{p-6}\tau+p(p-1)(p-2)(p-3)[a_1+a_2v^2-2(20+3p$$
$$\times(p-4))(b_1+b_2v^2)B^2]AB^4\tanh^{p-4}\tau-p(p-1)\Big\{k^2-v^2+[4(2+p(p-2))(a_1+a_2v^2)-(136+5p$$
$$\times(p-2)(23+3p(p-2)))(b_1+b_2v^2)B^2]B^2\Big\}AB^2\tanh^{p-2}\tau+2p^2\Big\{k^2-v^2+[(5+3p^2)(a_1+a_2v^2)-2(28$$
$$+5p^2(7+p^2))(b_1+b_2v^2)B^2]B^2\Big\}AB^2\tanh^{p}\tau-p(p+1)\Big\{k^2-v^2+[4(2+p(p+2))(a_1+a_2v^2)-(136$$
$$+5p(p+2)(23+3p(p+2)))(b_1+b_2v^2)B^2]B^2\Big\}AB^2\tanh^{p+2}\tau+p(p+1)(p+2)(p+3)[a_1+a_2v^2-2(20$$
$$+3p(p+4))(b_1+b_2v^2)B^2]AB^4\tanh^{p+4}\tau+p(p+1)(p+2)(p+3)(p+4)(p+5)\left(b_1+b_2v^2\right)AB^6\tanh^{p+6}\tau$$
$$-8cn^2p^2A^{2n}B^2\tanh^{2np}\tau+2cnp(2np-1)A^{2n}B^2\tanh^{2np-2}\tau+2cnp(2np+1)A^{2n}B^2\tanh^{2np+2}\tau=0, \quad (14)$$

where the argument was redefined as

$$\tau=B(x-vt) \quad (15)$$

for simplicity. A proper balance allows us to equate the exponents $2np=p+2$ leading to (4). Notice that the same value for the parameter p can be obtained by pairing $2np-2=p$ with p or $2np+2$ with $p+4$. Thus, from the stand-alone elements $\tanh^{p-6}\tau$

$$p=1, \quad (16)$$

which therefore implies that

$$n=\frac{3}{2}, \quad (17)$$

after equating (4) and (16). Such a value a of n slightly simplifies the system under study (1) to

$$q_{tt}-k^2q_{xx}+c\left(q^3\right)_{xx}+a_1q_{xxxx}+a_2q_{xxtt}+b_1q_{xxxxxx}+b_2q_{xxxxtt}=0. \quad (18)$$

Then, substituting (16) and (17) into (14), and setting the coefficients of the linearly independent functions $\tanh^j\tau$ for $j=1,3,5,7$ to zero yields the same expression for the velocity as in (7) and the amplitude-width relation

$$A=\sqrt{-\frac{(b_1+b_2k^2)+8(a_1b_2-a_2b_1)B^2}{3cb_2}} \quad (19)$$

whenever

$$cb_2\left\{\left(b_1+b_2k^2\right)+8\left(a_1b_2-a_2b_1\right)B^2\right\}<0. \quad (20)$$

In addition, the remaining two identities lead to

$$B=\sqrt{\frac{b_1+b_2k^2}{2(a_2b_1-a_1b_2)}}, \quad (21)$$

constrained by

$$\left(b_1+b_2k^2\right)\left(a_2b_1-a_1b_2\right)>0. \quad (22)$$

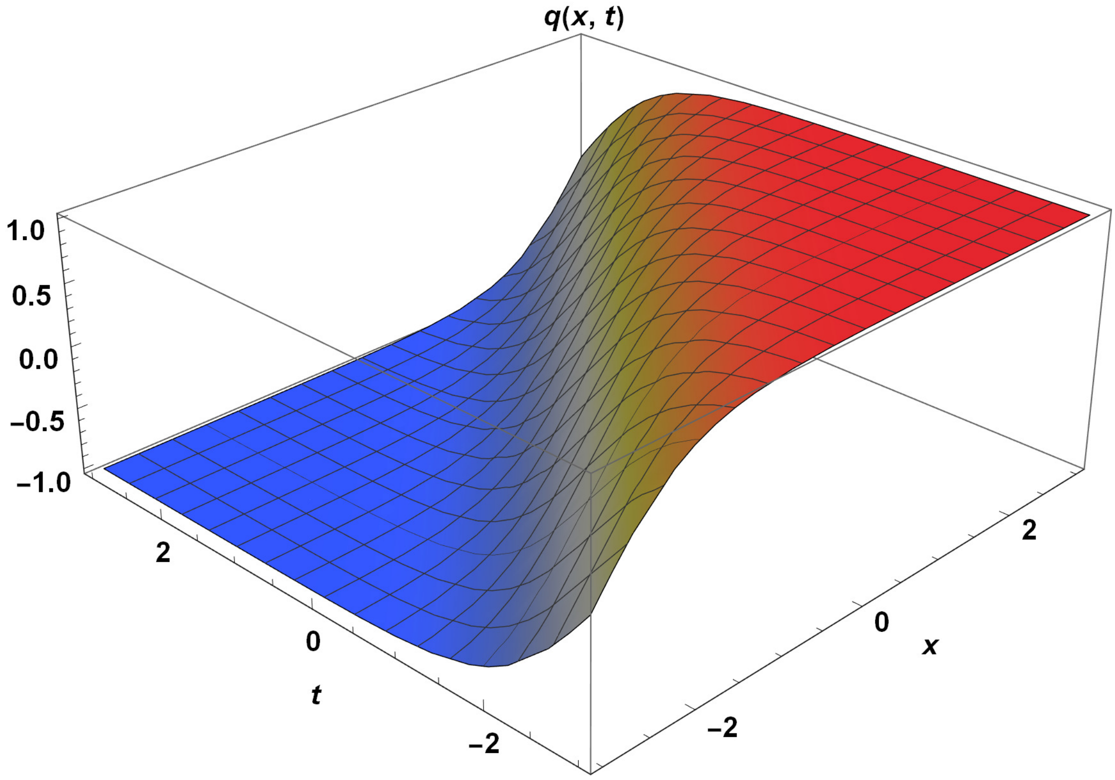

Fig. 2. Profile of a shock wave (25) with $b_1 = 1$, $b_2 = -2$, $k = 1$, $c = 1$, $a_1 = -1$ and $a_2 = 1$.

Thus, in view of (21), the amplitude (19) can be fully written in terms of the model parameter as

$$A = \sqrt{-\frac{(b_1 + b_2 k^2)}{c}} \tag{23}$$

as long as

$$c(b_1 + b_2 k^2) < 0. \tag{24}$$

Therefore, the shock wave solution to the slightly reduced system (18) is given by

$$q(x,t) = A \tanh[B(x - vt)], \tag{25}$$

where the amplitude is discussed in (23), the inverse width is given in (21), and as for solitary waves, the velocity is described in (7). The solvability conditions for the obtained shock waves are (8), (20), (22) and (24). Such conditions guarantee the existence of the shock wave solution. Figure 2 indicates the surface plot of a shock wave.

2.3. Singular solitary waves (type-I)

In this section we discuss the first type of singular solitary waves, namely the nonlinear waves described by the ansatz

$$q(x,t) = A \operatorname{csch}^p [B(x - vt)], \tag{26}$$

where, as in the previous two cases, A, B and v describe the amplitude, inverse width and wave velocity, respectively. The unknown parameter p will be determined by proper balance. Thus, the task is now to discover the parameters p, A, B and v so that, when substituting into (26), the resulting expression satisfies the BE (1). To proceed, we first substitute (26) into (1), disintegrating the latter into

$$-p^2[k^2-v^2-p^2(a_1+a_2v^2+p^2(b_1+b_2v^2)B^2)B^2]AB^2\operatorname{csch}^p\tau - p(p+1)\Big\{k^2-v^2-(2(2+p(2+p))$$
$$\times[a_1+a_2v^2]+(4+p(p+2))(4+3p(p+2))[b_1+b_2v^2]B^2)B^2\Big\}AB^2\operatorname{csch}^{p+2}\tau + p(p+1)(p+2)$$
$$\times(p+3)[a_1+a_2v^2+(20+3p(p+4))(b_1+b_2v^2)B^2]AB^4\operatorname{csch}^{p+4}\tau + p(p+1)(p+2)(p+3)(p+4)$$
$$\times(p+5)(b_1+b_2v^2)AB^6\operatorname{csch}^{p+6}\tau + 2cnp(2np+1)A^{2n}B^2\operatorname{csch}^{2np+2}\tau + 4cn^2p^2A^{2n}B^2\operatorname{csch}^{2np}\tau = 0 \quad (27)$$

after simplification. Once again, the notation $\tau = B(x-vt)$ has been adopted only for simplicity. With the aid of the balancing principle one can equate

$$2np+2 = p+4 \qquad \text{or} \qquad 2np = p+2, \tag{28}$$

which allows us to retrieve (4). Now, substituting the obtained value for the parameter p into (27) and setting the coefficients of the linearly independent functions $\operatorname{csch}^{\frac{j}{2n-1}}\tau$ to zero for $j = 2, 4n, 8n-2$, and $12n-4$, one gets the velocity as in (7) and the parameter

$$A = \left[-\frac{(2n+1)(b_1+b_2k^2)}{2cb_2}\right]^{\frac{1}{2n-1}} \tag{29}$$

as long as

$$cb_2(2n+1)\left(b_1+b_2k^2\right) < 0. \tag{30}$$

Similarly, the inverse width proved to be as in (9) after considering (7). Therefore, the singular solitary type-I wave solution to the BE with the presence of surface tension (1) is given by

$$q(x,t) = A\operatorname{csch}^{\frac{2}{2n-1}}[B(x-vt)], \tag{31}$$

where the corresponding amplitude A, the inverse width B and the wave velocity v are given in (29), (9) and (7), respectively. For the resulting solution to exist, the contraints (8), (10) and (30) have to be satisfied.

2.4. Singular solitary waves (type-II)

The last type of wave to be considered in this work is the second type of singular solitary waves, whose ansatz is given by

$$q(x,t) = A\coth^p[B(x-vt)], \tag{32}$$

where the parameters A, B, v and the exponent p have the same meaning as for the three wave forms discussed in the previous three subsections. For convenience, we adopt again the notation $\tau = B(x-vt)$. Now, the substitution of (32) into (1) gives

$$p(p-1)(p-2)(p-3)(p-4)(p-5)(b_1+b_2v^2)AB^6\coth^{p-6}\tau+p(p-1)(p-2)(p-3)[a_1+a_2v^2-2(20+3p \times(p-4))(b_1+b_2v^2)B^2]AB^4\coth^{p-4}\tau-p(p-1)\Big\{k^2-v^2+[4(2+p(p-2))(a_1+a_2v^2)-(136+5p(p-2) \times(23+3p(p-2)))(b_1+b_2v^2)B^2]B^2\Big\}AB^2\coth^{p-2}\tau+2p^2\Big\{k^2-v^2+[(5+3p^2)(a_1+a_2v^2)-2(28+5p^2 \times(7+p^2))(b_1+b_2v^2)B^2]B^2\Big\}AB^2\coth^{p}\tau-p(p+1)\Big\{k^2-v^2+[4(2+p(p+2))(a_1+a_2v^2)-(136+5p \times(p+2)(23+3p(p+2)))(b_1+b_2v^2)B^2]B^2\Big\}AB^2\coth^{p+2}\tau+p(p+1)(p+2)(p+3)[a_1+a_2v^2-2(20+3p \times(p+4))(b_1+b_2v^2)B^2]AB^4\coth^{p+4}\tau+p(p+1)(p+2)(p+3)(p+4)(p+5)(b_1+b_2v^2)AB^6\coth^{p+6}\tau -8cn^2p^2A^{2n}B^2\tanh^{2np}\tau+2cnp(2np-1)A^{2n}B^2\coth^{2np-2}\tau+2cnp(2np+1)A^{2n}B^2\coth^{2np+2}\tau=0 \quad (33)$$

after simplification. A proper balance yields again the value of p given in (4), while the stand-alone elements $\tanh^{p\pm 6}\tau$ yield (16). Comparing both resulting values of p leads to (17). Substituting (16) and (17) into (33) results in identical results as for shock waves, namely (18)–(24) along with corresponding constraints. Therefore, the type-II singular solitary wave solution for the system (18) has the form

$$q(x,t)=A\coth[B(x-vt)], \quad (34)$$

where the corresponding amplitude, inverse width and velocity, along with corresponding solvability conditions, are the same as for the shock waves discussed above.

3. LIE SYMMETRY ANALYSIS

The Lie symmetry analysis is one of the most powerful mathematical tools to handle a nonlinear differential equation from its integrability standpoint [1,2,11]. This methodology has been successfully implemented in several models [12–14]. In this section, we try to construct symmetries, symmetry reductions and group invariant solutions to the BE in the presence of surface tension (1) via the classical Lie method [11–13]. For this, let us consider the Lie group of point transformations as

$$\begin{aligned} q^* &= q+\varepsilon\eta(x,t,q)+O(\varepsilon^2),\\ x^* &= x+\varepsilon\xi(x,t,q)+O(\varepsilon^2),\\ t^* &= t+\varepsilon\tau(x,t,q)+O(\varepsilon^2), \end{aligned} \quad (35)$$

which leaves the equation (1) invariant. The method for determining the symmetry group of (1) consists in finding the infinitesimals η,ξ and τ, which are functions of x,t,q. Assuming that the system is invariant under the transformations (35), we get the following relations from the first-order coefficients ε:

$$\begin{aligned} &\eta^{tt}-k^2\eta^{xx}+c(2n)(2n-1)(2q_xq^{2n-2}\eta^x+(2n-2)\eta q^{2n-3}{q_x}^2)+2nc(q^{2n-1}\eta^{xx}+(2n-1)\eta q^{2n-2}q_{xx})\\ &+a_1\eta^{xxxx}+a_2\eta^{xxtt}+b_1\eta^{xxxxxx}+b_2\eta^{xxxxtt}=0, \end{aligned} \quad (36)$$

where $\eta^{x}, \eta^{xx}, \eta^{tt}, \eta^{xxxx}, \eta^{xxtt}, \eta^{xxxxxx}, \eta^{xxxxtt}$ are extended (prolonged) infinitesimals acting on an enlarged space that includes all derivatives of the dependent variables $q_x, q_{xx}, q_{tt}, q_{xxxx}, q_{xxtt}, q_{xxxxxx}$ and q_{xxxxtt}. The infinitesimals are determined from the invariance conditions (36) by setting the coefficients of different differentials equal to zero. We obtain a large number of partial differential equations (PDEs) in η, ξ and τ that need to be satisfied. After long and tedious calculations, the general solution to this large system provides us with the following forms for the infinitesimal elements η, ξ and τ:

$$\xi = C_1,\ \tau = C_2,\ \eta = 0, \tag{37}$$

where C_1 and C_2 are arbitrary constants and the above symmetries reduce the equation (1) to an ODE using the characteristic equation

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{dq}{\eta}. \tag{38}$$

Using the symmetries obtained in (37), we get the similarity variables and the traveling wave transformation as

$$q(x,t) = F(\zeta),\ \zeta = B(x - vt), \tag{39}$$

which enables us to reach solitons similar to those obtained by assuming the ansatzes (2), (13), (26) and (32).

However, in one more case, when $n = \frac{1}{2}$, we get some different symmetries, which have been mentioned as

$$\xi = C_1,\ \tau = C_2,\ \eta = C_3 q + C_4, \tag{40}$$

where C_1, C_2, C_3, C_4 are arbitrary constants. Using the symmetries obtained above and the characteristic equation (38), we obtain the following similarity variables and transformation:

$$q(x,t) = e^{t} F(\zeta),\ \zeta = x - wt, \tag{41}$$

where w is the velocity of the traveling wave. On substituting (41) into the equation (1), we get the reduced ODE, in this case as

$$\begin{aligned} &F(\zeta) - 2wF'(\zeta) + (w^2 - k^2 + c + a_2)F''(\zeta) - 2a_2 wF'''(\zeta) + (a_2 w^2 + a_1 + b_2)F^{iv}(\zeta) - 2wb_2 F^{v}(\zeta) \\ &+ (b_1 + w^2 b_2)F^{vi}(\zeta) = 0. \end{aligned} \tag{42}$$

Due to the complexity of the reduced ODE above, we are able to recover the only trivial solution of the equation.

4. CONSERVATION LAWS

We present some preliminaries on conservation laws, which will be used in the following analyses.

For an rth-order system of PDEs of n independent variables and m dependent variables $u = (u_1, u_2, \ldots, u_m)$, viz., $E^{\mu}(\underline{x}, u, u_{(1)}, \ldots, u_{(r)}) = 0$, where $u, u_1, \ldots, u_k$ denote the collections of all first- , second- , etc., order derivatives of u, and with the total differentiation operator with respect to x^i given by $D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha} \frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha} \frac{\partial}{\partial u_j^{\alpha}} + \ldots$, we define the conservation law to be the closed form $D_{x_1} T^{x_1} + D_{x_2} T^{x_2} + \ldots = 0$ on the solutions of the system of PDEs. We may resort to the multiplier approach based on the well-known result that the Euler–Lagrange operator annihilates the total divergence [1]. That is, if there exists nontrivial differential functions Q^{μ}, called a 'multiplier', such that

$$Q^{\mu}(\underline{\mathrm{x}},u,u_{(1)}\dots)E^{\mu}(\underline{\mathrm{x}},u,u_{(1)},\dots,u_{(r)})=D_{x_1}T^{x_1}+D_{x_2}T^{x_2}+\dots, \tag{43}$$

for some (conserved) vector $(T^{x_1},T^{x_2},\dots)$, then

$$\frac{\delta}{\delta u}Q(\underline{\mathrm{x}},u,u_{(1)}\dots)E(\underline{\mathrm{x}},u,u_{(1)},\dots,u_{(r)})=0, \tag{44}$$

where $\frac{\delta}{\delta u}$ is the Euler operator. This equation can be used to construct the Qs, and some procedure may then be applied to derive the respective conserved flows. We note that the conserved vectors may be expressed, alternatively, as differential 'forms' [2].

The conservation laws are divided into two subsections based on arbitrary n and when $n=1$.

4.1. Case-I (arbitrary *n*)

The multiplier approach leads to four nontrivial multipliers for the general case on n, viz.,

$$Q_A=1, \qquad Q_B=x, \qquad Q_C=t, \qquad Q_D=xt. \tag{45}$$

None of these include momentum or energy conservation. In some cases below, we only present the densities T^t:

$$\begin{aligned}
&A. \quad Q=1:\\
&T_1^t=-\tfrac{1}{3}b_2q_{xxxxt}-\tfrac{1}{2}a_2q_{xxt}-q_t,\\
&T_1^x=-\tfrac{2}{3}b_2q_{xxxtt}-b_1q_{xxxx}-\tfrac{1}{2}a_2q_{xtt}-a_1q_{xxx}+k^2q_x-ncq^{n-1}q_x.\\
&B. \quad Q=x:\\
&T_2^t=-\tfrac{1}{3}b_2xq_{xxxxt}+\tfrac{4}{15}b_2q_{xxxt}-\tfrac{1}{2}a_2xq_{xxt}+\tfrac{1}{3}a_2q_{xt}a_2-xq_t,\\
&T_2^x=-nxcq_xq^{n-1}+cq^n-\tfrac{2}{3}b_2xq_{xxxtt}-b_1xq_{xxxx}+\tfrac{2}{5}b_2q_{xxtt}\\
&+b_1q_{xxx}-\tfrac{1}{2}a_2xq_{xtt}-a_1xq_{xxx}+\tfrac{1}{6}a_2q_{tt}+a_1q_{xx}+xk^2q_x-k^2q.\\
&C. \quad Q=t:\\
&T_3^t=-\tfrac{1}{3}b_2tq_{xxxxt}+\tfrac{1}{15}b_2q_{xxx}-\tfrac{1}{2}a_2tq_{xxt}+\tfrac{1}{6}a_2q_{xx}-tq_t+q,\\
&T_3^x=-\tfrac{2}{3}b_2tq_{xxxtt}-b_1tq_{xxxx}+\tfrac{4}{15}b_2q_{xxxt}-\tfrac{1}{2}a_2tq_{xtt}-a_1tq_{xxx}+\tfrac{1}{3}a_2q_{xt}+tk^2q_x-nctq_xq^{n-1}.\\
&D. \quad Q=xt:\\
&T_4^t=-\tfrac{1}{3}b_2xtq_{xxxxt}+\tfrac{4}{15}b_2tq_{xxxt}+\tfrac{1}{15}b_2xq_{xxx}-\tfrac{1}{2}a_2xtq_{xxt}-\tfrac{2}{15}b_2q_{xxx}+\tfrac{1}{3}a_2tq_{xt}\\
&+\tfrac{1}{6}a_2xq_{xx}-xtq_t-\tfrac{1}{3}a_2q_x+q.
\end{aligned} \tag{46}$$

4.2. Case-II ($n = 1$)

For $n = 1$, we obtain infinitely many conservation laws, which include higher-order laws that originate from higher-order multipliers. Here, energy (E) and linear momentum (M) are conserved:

a. $Q_a = q_x$ (linear momentum):

$$\begin{aligned} T_1^t = & -\tfrac{3}{10}b_2 q_{xxxxt}q_x + \tfrac{7}{30}b_2 q_{xxxt}q_{xx} + \tfrac{1}{10}b_2 q_{xxx}q_{xt} - \tfrac{1}{6}b_2 q_{xxx}q_{xxt} - \tfrac{5}{12}a_2 q_{xxt}q_x + \tfrac{1}{4}a_2 q_{xt}q_{xx} \\ & -\tfrac{1}{30}b_2 q_t q_{xxxx} - \tfrac{1}{12}a_2 q_t q_{xxx} - \tfrac{1}{2}q_t q_x + \tfrac{1}{6}b_2 q q_{xxxxxt} + \tfrac{1}{4}a_2 q q_{xxxt} + \tfrac{1}{2}q q_{xt}. \end{aligned} \tag{47}$$

b. $Q_b = q_t$ (energy):

$$\begin{aligned} T_2^t = & -\tfrac{1}{5}b_2 q_t q_{xxxxt} + \tfrac{1}{5}b_2 q_{xxxt}q_{xt} + \tfrac{1}{30}b_2 q_{tt}q_{xxx} - \tfrac{1}{10}b_2 q_{xxt}^2 - \tfrac{1}{3}a_2 q_t q_{xxt} - \tfrac{1}{15}b_2 q_{xtt}q_{xxx} \\ & +\tfrac{1}{6}a_2 q_{xt}^2 + \tfrac{1}{10}b_2 q_{xxtt}q_{xx} + \tfrac{1}{12}a_2 q_{tt}q_{xx} - \tfrac{1}{2}q_t^2 - \tfrac{2}{15}b_2 q_{xxxtt}q_x - \tfrac{1}{6}a_2 q_{xtt}q_x - \tfrac{1}{3}b_2 q q_{xxxxtt} \\ & -\tfrac{1}{4}a_2 q q_{xxtt} + \tfrac{1}{2}k^2 q q_{xx} - \tfrac{1}{2}a_1 q q_{xxx} - \tfrac{1}{2}b_1 q q_{xxxxxx} - \tfrac{1}{2}c q q_{xx}. \end{aligned} \tag{48}$$

An example of a higher-order density is

c. $Q_c = q_{xxt}$:

$$\begin{aligned} T_3^t = & \tfrac{1}{15}b_2 q_{xt}q_{xxxxxt} + \tfrac{1}{30}b_2 q_x q_{xxxxxtt} + \tfrac{1}{6}c q_x q_{xxx} - \tfrac{1}{30}b_2 q_t q_{xxxxxxt} + \tfrac{1}{6}b_1 q_x q_{xxxxxxx} \\ & -\tfrac{1}{6}b_1 q_{xx}q_{xxxxxx} + \tfrac{1}{30}b_2 q_{xxx}q_{xxtt} + \tfrac{1}{6}a_2 q_{xt}q_{xxxt} - \tfrac{1}{6}k^2 q_x q_{xxx} + \tfrac{1}{3}q q_{xxtt} - \tfrac{1}{6}c q_{xx}^2 \\ & +\tfrac{1}{6}k^2 q_{xx}^2 - \tfrac{1}{6}q_{xx}q_{tt} - \tfrac{1}{4}a_2 q_{xxt}^2 - \tfrac{1}{2}q_t q_{xxt} + \tfrac{1}{6}q_x q_{xtt} + \tfrac{1}{6}a_1 q_x q_{xxxx} - \tfrac{1}{12}a_2 q_t q_{xxxxt} \\ & -\tfrac{4}{15}b_2 q_{xxxxt}q_{xxt} - \tfrac{1}{12}a_2 q_{xx}q_{xxtt} - \tfrac{1}{15}b_2 q_{xx}q_{xxxxtt} + \tfrac{1}{6}k^2 q q_{xxx} - \tfrac{1}{6}c q q_{xxx} - \tfrac{1}{6}a_1 q_{xx}q_{xxx} \\ & -\tfrac{1}{15}b_2 q_{xxx}q_{xxxtt} - \tfrac{1}{6}b_1 q q_{xxxxxxxx} + \tfrac{2}{15}b_2 q_{xxxt}^2 - \tfrac{1}{6}a_1 q q_{xxxxxx} + \tfrac{1}{12}a_2 q q_{xxxxtt}. \end{aligned} \tag{49}$$

5. CONSERVED QUANTITIES

The conserved densities derived in the previous section lead to the conserved quantities based on the solitary wave given by (12). The next two subsections are based on the two cases for arbitrary n and $n = 1$, respectively. The results of this section are reported for the first time. The previously reported work on conservation laws was rendered incomplete since it never computed conserved quantities from the densities, not to mention the general power law of nonlinearity [8].

5.1. Case-I (arbitrary n)

The conserved quantities, upon using the solitary wave solution given by (12) are, respectively, given as

$$I_1 = \int_{-\infty}^{\infty} T_1^t dx = 0, \tag{50}$$

$$I_2 = \int_{-\infty}^{\infty} T_2^t dx = \int_{-\infty}^{\infty} q dx = \frac{A}{B}\frac{\Gamma\left(\frac{1}{2n-1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2n-1}+\frac{1}{2}\right)}, \tag{51}$$

$$I_3 = \int_{-\infty}^{\infty} T_3^t dx = \int_{-\infty}^{\infty} q dx = \frac{A}{B}\frac{\Gamma\left(\frac{1}{2n-1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2n-1}+\frac{1}{2}\right)}, \tag{52}$$

and

$$I_4=\int_{-\infty}^{\infty}T_4^t dx=\int_{-\infty}^{\infty}(1-vt)q dx=\frac{(1-vt)A}{B}\frac{\Gamma\left(\frac{1}{2n-1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2n-1}+\frac{1}{2}\right)}. \tag{53}$$

For I_4 to be a conserved quantity, it is necessary to have

$$\frac{dI_4}{dt}=0, \tag{54}$$

which prompts

$$v=0. \tag{55}$$

Thus, this conserved quantity is valid for a stationary solitary wave and is given by

$$I_4=\frac{A}{B}\frac{\Gamma\left(\frac{1}{2n-1}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2n-1}+\frac{1}{2}\right)}. \tag{56}$$

5.2. Case-II ($n=1$)

For $n=1$, the solitary wave solution given by (12) is reduced to

$$q(x,t)=A\,\mathrm{sech}^2[B(x-vt)]. \tag{57}$$

Thus, the conservation laws are given by

$$I_1=M=\int_{-\infty}^{\infty}T_1^t dx=\frac{v}{10}\int_{-\infty}^{\infty}\left(7b_2q_{xxx}^2+10a_2q_{xx}^2+10q_x^2\right)dx=\frac{vA^2B}{525}\left(560+1600a_2B^2+224b_2B^4\right), \tag{58}$$

$$\begin{aligned} I_2=E &= \int_{-\infty}^{\infty}T_2^t dx=\frac{1}{30}\int_{-\infty}^{\infty}\left\{\left(15b_1-2b_2v^2\right)q_{xxx}^2+15a_2v^2q_{xx}^2+15\left(c^2-v^2-k^2\right)q_x^2\right\}dx \\ &= \frac{8A^2B}{1575}\left[4B^4\left(15b_1-2b_2v^2\right)+300a_2v^2B^2+105\left(c-v^2-k^2\right)\right], \end{aligned} \tag{59}$$

and

$$I_3=\int_{-\infty}^{\infty}T_3^t dx=\frac{32A^2B^3}{17,325}\left[39\left(14b_2v^2-15b_1\right)B^4+55\left(a_1-3a_2v^2\right)B^2-275\left(2c+2v^3+4v^2-v\right)\right]. \tag{60}$$

6. CONCLUSIONS

The current work has recovered solitary waves, shock waves and singular solitary waves for shallow water waves, with the inclusion of the surface tension effect along with conservation laws. Thus, with the fundamental results a deluge of opportunities has emerged for the model. A future agenda is to study the model numerically by implementing the Laplace Adomian decomposition approach or the variational iteration approach. Additional integration schemes will be implemented for the 6BE to provide fresher perspectives [14,15]. In addition, the soliton perturbation theory would be applied to obtain the adiabatic parameter dynamics of solitary waves. The effect of randomness also needs to be triggered so that the 6BE can be studied with additive as well as multiplicative noise with the aid of Ito and/or Stratonovich calculus. These are only the tip of the iceberg!

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