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Free vibration analysis of tapered Timoshenko beam with higher order Haar wavelet method

*This paper is dedicated to the 100th birthday of Professor Ülo Lepik.
This study can be considered as a continuation of the research in the area of
Haar wavelet methods started by Professor Ülo Lepik*

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Abstract. In the current study, the higher order Haar wavelet method based formulation is developed for the analysis of the free vibrations of the tapered Timoshenko beam. The clamped-clamped and clamped-pinned boundary conditions are explored and the results with the 4th order and the 6th order of convergence are presented. The results are found to be in good agreement with the corresponding results of the Ritz method. The proposed approach can be considered as the principal improvement of the widely used Haar wavelet method providing the same accuracy with the several magnitudes lower mesh. Thus, the higher order Haar wavelet method has reduced the computational cost in comparison with the widely used Haar wavelet method since the computational complexity of both methods is determined by the mesh used. In the case of the fixed equal mesh used for both methods, the higher order Haar wavelet method results in the several magnitudes lower absolute error without a remarkable increase in computational complexity. The cost needed to pay for higher accuracy is hidden in a certain increase in the implementation complexity compared with the widely used Haar wavelet method.

Key words: higher order Haar wavelet method, tapered Timoshenko beam, free vibration.

1. INTRODUCTION

Development and adaptation of computational methods and mathematical modelling techniques are rapidly evolving research areas with the main focus on finding more accurate, less time-consuming, and simpler approximations.

The Haar wavelet method (HWM) was first introduced in [1–2]. According to Chen and Hsiao's approach, the highest order of derivatives included in a differential equation is expanded into a series of Haar functions [1–2]. This method is applied to solving differential and integro-differential equations covering applications in various research areas such as engineering, natural sciences, etc. [3–9]. Furthermore, this method is used as a numerical solution to linear and nonlinear delay differential equations [10], and space derivatives are obtained through the Haar wavelet collocation method to solve 1D and 2D cubic nonlinear Schrodinger equations [11]. In [12] the accuracy and convergence results of the HWM are presented. Based on the obtained

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results, it can be concluded that despite its simple implementation, the HWM needs refinement in order to compete with such widely used numerical methods as the finite difference method and the differential quadrature method.

Recently, the higher order Haar wavelet method (HOHWM) was introduced in [13] in order to improve the accuracy and convergence of the previously proposed Haar wavelet method. The HOHWM has been applied with success to solving differential equations, vibration, and buckling response of beams [14–18]. Theoretical and numerical analyses of the free and forced vibration of homogeneous and functionally graded Timoshenko beams have been performed [19–22]. In the case of tapered beams, many approaches have been used for analysing the Timoshenko beam that has a non-uniform cross-section [23–26].

The HOHWM is applied with success to the analysis of plate and shell structures using Euler–Bernoulli and zig-zag theories. In this paper the HOHWM approach is adapted to the Timoshenko beam theory.

2. HOHWM APPROACH TO FREE VIBRATION ANALYSIS OF THE TIMOSHENKO BEAM

In this section, the formulation of the free vibration of the tapered Timoshenko beam and boundary conditions are introduced.

2.1. Free vibration of the Timoshenko beam

A schematic view of the Timoshenko beam with a non-uniform cross-section along the length, *x*-direction, is shown in Fig. 1.

Herein, free vibration of homogeneous tapered Timoshenko beams has been investigated. The material properties of the beams are assumed to be constant. Firstly, the cross-sectional area $A(x)$ and the moment of inertia $I(x)$ are presented as

$$A(x) = A_0\left(1-\frac{cx}{L}\right), I(x) = I_0\left(1-\frac{cx}{L}\right)^3, \quad x \in [0, L], \tag{1}$$

where A_0 and I_0 are the area and the moment of inertia at the base of the beam, respectively. L is the length of the beam, E denotes Young's modulus, G refers to shear modulus, ρ represents mass density, and k is the shear correction factor which is chosen to be 5/6. For the described Timoshenko beam, the basic governing differential equations for transverse vibration of the tapered beam can be presented as

$$\begin{gathered} \frac{\partial}{\partial x}\left(EI(x)\frac{\partial \varphi}{\partial x}\right) + \kappa GA(x)\left(\frac{\partial w}{\partial x} - \varphi\right) - \rho I\frac{\partial^2 \varphi}{\partial t^2} = 0 \\ \frac{\partial}{\partial x}\left[\kappa GA(x)\left(\frac{\partial w}{\partial x} - \varphi\right)\right] - \rho A\frac{\partial^2 w}{\partial t^2} = 0 \end{gathered}, \tag{2}$$

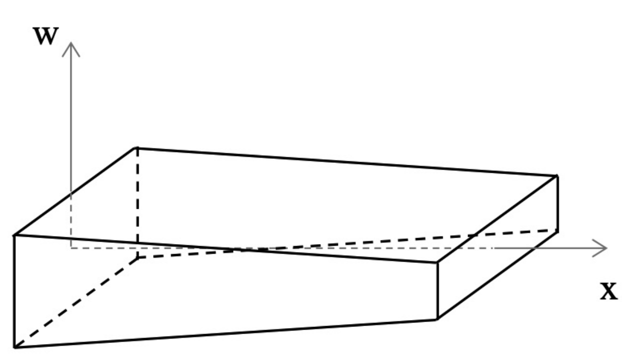

Fig. 1. Schematic view of a tapered beam.

where w and φ are the transverse deflection and rotation of the cross-section, respectively. The bending moment M and the shear force Q at any cross-section can be read as

$$M = EI(x)\frac{\partial \varphi}{\partial x}, \quad Q = kGA(x)\left(\frac{\partial w}{\partial x} + \varphi\right). \tag{3}$$

The boundary conditions for the beam can be expressed

$$\begin{aligned} &\text{for the clamped edge as: } w = 0, \quad \varphi = 0, \\ &\text{for the pinned edge as: } w = 0, \quad M = 0. \end{aligned} \tag{4}$$

2.2. Higher order Haar wavelet method

The higher order Haar wavelet method (HOHWM) is developed as an improvement of the widely used Haar wavelet method (HWM) [13].

The n-th order ordinary differential equation, in general, can be presented as

$$G(x, u, u', u'', \dots u^{(n-1)}, u^{(n)}) = 0, \tag{5}$$

where n represents the order of the highest derivative involved in the differential equation. In the HOHWM, in comparison to the Haar wavelet method, the order of expansion is increased by $2s$, Eq. (6). Based on the Haar wavelet, the expansion is presented as

$$f(x) = \frac{d^{n+2s}u(x)}{dx^{n+2s}} = \sum_{i=1}^{\infty} a_i h_i(x), \, s = 1,2, \dots, \tag{6}$$

in which $h_i(x)$ is the Haar function [18]

$$h_i(x) = \begin{cases} 1 & \text{for} \quad x \in [\xi_1(i), \xi_2(i)) \\ -1 & \text{for} \quad x \in [\xi_2(i), \xi_3(i)), \\ 0 & \text{elsewhere} \end{cases} \tag{7}$$

where $i = m + k + 1$, $m = 2^j$ is a maximum number of square waves arranged in the interval $[A,B]$ and the parameter k indicates the location of the particular square wave [18]

$$\begin{gathered} \xi_1(i) = A + 2k\mu\Delta x, \qquad \xi_2(i) = A + (2k+1)\mu\Delta x, \qquad \xi_3(i) = A + 2(k+1)\mu\Delta x, \\ \mu = M/m, \qquad \Delta x = (B-A)/(2M). \end{gathered} \tag{8}$$

The integrals of the Haar functions (7) of order n can be expressed as [13]

$$p_{n,i}(x) = \begin{cases} 0 & & x \in [A, \xi_1(i)) \\ \frac{(x-\xi_1(i))^n}{n!} & & x \in [\xi_1(i), \xi_2(i)) \\ \frac{(x-\xi_1(i))^n - 2(x-\xi_2(i))^n}{n!} & for & x \in [\xi_2(i), \xi_3(i)) . \\ \frac{(x-\xi_1(i))^n - 2(x-\xi_2(i))^n + (x-\xi_3(i))^n}{n!} & & x \in [\xi_3(i), B) \\ 0 & & elsewhere \end{cases} \tag{9}$$

The differential equation can be satisfied in selected uniform grid points

$$x_{iL} = \frac{i}{2M}, \; x_{iR} = 1 - \frac{i}{2M}, i = 0, \dots, s-1, \tag{10}$$

where L and R are the added collocation points on the left and right boundary, respectively. Then the numerical order of the convergence of the method can be estimated by

$$\text{Convergence rate} = \frac{\log\left(\dfrac{F_{i-1} - F_{Ref}}{F_i - F_{Ref}}\right)}{\log(2)}, \tag{11}$$

where F_{Ref} is the existing solution, which in the current solution is obtained from the Ritz method [21].

3. NUMERICAL RESULTS

In order to showcase the accuracy of the formulation proposed above, the values of natural frequencies of the Timoshenko beam under two arbitrary boundary conditions are presented. Table 1 presents the effect of

Table 1. Effect of taper ratio on non-dimensional natural frequencies of the C-C Timoshenko beam

	N	HWM			HOHWM 4th			HOHWM 6th		
		Frequency	A. error	Conv. rate	Frequency	A. error	Conv. rate	Frequency	A. error	Conv. rate
c = 0	4	13.96275845	1.28e-01		13.84197845	7.22e-03		13.83477975	2.13e-05	
	8	13.86655612	3.18e-02	2.0091	13.83519214	4.34e-04	4.0576	13.83476001	1.56e-06	5.4358
	16	13.84269132	7.93e-03	2.003	13.83478525	2.68e-05	4.0146	13.83475854	8.43e-08	5.7522
	32	13.83674066	1.98e-03	2.0007	13.83476012	1.67e-06	4.0036	13.83475846	1.51e-09	5.9467
	64	13.83525392	4.95e-04	2.00019	13.83475856	1.04e-07	4.0009	13.83475846	7.71e-10	5.9955
	128	13.83488232	1.24e-04	2.00004	13.83475846	6.53e-09	4.0002	13.83475845	4.08e-11	5.9999
	256	13.83478942	3.10e-05	2.00001	13.83475845	4.08e-10	4.0000	13.83475845	1.40e-13	6.0000
	Existing result = 13.834758									
c = 0.4	4	13.38213007	9.60E-01		12.42216412	3.56E-02		12.42216313	4.40E-03	
	8	12.51071171	8.85E-02	2.0154	12.45779287	7.62E-03	4.3451	12.42655937	7.71E-04	6.0527
	16	12.43177774	9.61E-03	2.0095	12.42293499	7.72E-04	4.0623	12.42217245	9.32E-06	6.0129
	32	12.42292193	7.59E-04	2.0037	12.42221943	5.63E-05	4.0103	12.42216313	4.41E-07	6.0099
	64	12.42223492	7.18E-05	2.0018	12.42978489	9.92E-07	4.0096	12.42293432	4.36E-09	6.0042
	128	12.42218071	1.76E-05	2.0008	12.42216317	4.02E-08	4.0073	12.42216357	7.28E-10	6.0017
	256	12.42216626	3.13E-06	2.0003	12.42216313	4.39E-09	4.0023	12.42216313	2.71E-11	6.0009
	Existing result = 12.422163									
c = 0.8	4	10.7701461	1.04E+00		9.738846102	1.17E-02		9.727997702	8.52E-04	
	8	10.2871461	5.60E-01	2.0994	9.727886102	7.40E-04	4.0807	9.727181202	3.51E-05	6.0698
	16	9.782096102	5.50E-02	2.0848	9.727219602	7.35E-05	4.0713	9.727147068	9.66E-07	6.0695
	32	9.728230102	1.08E-03	2.0631	9.727157102	1.10E-05	4.0466	9.727146185	8.28E-08	6.0606
	64	9.727253202	1.07E-04	2.0480	9.727151332	5.23E-06	4.0402	9.727146103	6.74E-10	6.0326
	128	9.727156532	1.04E-05	2.0279	9.727146886	7.84E-07	4.0198	9.727146102	4.75E-11	6.0050
	256	9.727147652	1.55E-06	2.0051	9.727146151	4.90E-08	4.0074	9.727146102	6.84E-12	6.0007
	Existing result = 9.727146									

A. error – Absolute error

Table 2. Effect of boundary conditions on non-dimensional natural frequencies of the tapered Timoshenko beam (c = 0.2)

	N	HWM			HOHWM 4th			HOHWM 6th		
		Frequency	A. error	Conv. rate	Frequency	A. error	Conv. rate	Frequency	A. error	Conv. rate
C-P	4	12.48688739	1.80E+00		11.05355589	3.67E-01		10.77030682	8.34E-02	
	8	11.05808739	3.70E-01	2.3120	10.73676607	6.99E-02	4.8521	10.68746912	5.82E-04	7.6790
	16	10.75601079	6.91E-02	2.0624	10.6877406	8.53E-03	4.0629	10.68692751	4.01E-05	6.6293
	32	10.69501206	8.12E-03	2.0039	10.6869284	4.10E-04	4.0132	10.68688961	2.22E-06	6.0872
	64	10.68781264	9.25E-04	2.0007	10.68688976	2.37E-05	4.0083	10.6868877	3.06E-07	6.0034
	128	10.68715151	2.64E-04	2.0003	10.68688778	3.94E-06	4.0019	10.6868874	9.28E-09	6.0012
	256	10.68695081	6.30E-05	2.00002	10.6868874	7.31E-07	4.0005	10.68688739	2.79E-10	6.0008
	Existing result = 10.68689									
C-C	4	14.32226684	1.10E+00		13.26456684	4.23E-02		13.22543684	3.17E-03	
	8	13.95226684	7.30E-01	2.0906	13.22523684	2.97E-03	4.1523	13.22233484	6.80E-05	6.0228
	16	13.28756684	6.53E-02	2.0746	13.22246484	1.98E-04	4.0945	13.22226727	4.35E-07	6.0197
	32	13.22697684	4.71E-03	2.0595	13.22227401	7.17E-06	4.0840	13.22226686	1.76E-08	6.0131
	64	13.22245884	1.92E-04	2.0164	13.22226732	4.83E-07	4.0570	13.22226684	6.23E-09	6.0109
	128	13.22232314	5.63E-05	2.0117	13.22226737	5.34E-07	4.0338	13.22226684	6.31E-10	6.0072
	256	13.22227227	5.43E-06	2.0021	13.22226684	4.85E-09	4.0013	13.22226684	5.02E-11	6.0020
	Existing result = 13.222267									

taper ratio (c) for the beam under clamped-clamped (C-C) boundary conditions. The results are compared with the existing results obtained from the Ritz method and alternative methods employed in [22,25].

As expected, for the beam with the taper ratio other than $c = 0$, the non-dimensional natural frequency decreases for the higher value of c. Moreover, as it can be observed, the results of the higher order Haar wavelet method prove that in the case of the 4th and the 6th order of convergence, the absolute error reduces much faster by increasing the number of terms in the Haar wavelet method. This matter could be essential in the case of more complex problems, thus the accurate result can be obtained faster and with a smaller number of terms.

The effect of boundary conditions is shown in Table 2. For the tapered Timoshenko beam (c = 0.2), the results of two boundary conditions – clamped-clamped (C-C) and clamped-pinned (C-P) – are produced, which prove the above-mentioned point for the higher order Haar wavelet method. In the future study, the HOHWM is planned to be applied to design optimization of plate and shell structures [27–31].

4. CONCLUSIONS

During the last two years, the HOHWM has been applied with success to the analysis of plate and shell structures by using Euler–Bernoulli and zig-zag theories. In the current study, the HOHWM is extended to the vibration analysis of Timoshenko beams. The solution has been used to analyse the beam under two boundary conditions, clamped-clamped and clamped-pinned. The results for beams with different taper ratios prove that the higher order Haar wavelet method is accurate, and for the versions with the higher order of convergence (4th and 6th order) the absolute error drops extremely fast. These results can be translated to a faster, simpler, and more accurate solution for other structural analyses where the analytical solution is difficult to obtain.

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Muutuva ristlõikega Timoshenko tala vabavõnkumiste analüüs kõrgemat järku Haari lainikute meetodi abil

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Viimase kahe aasta jooksul on rakendatud kõrgemat järku Haari lainikute meetodit plaatide ja koorikute analüüsiks, kasutades peamiselt Euler-Bernoulli teooriat, ühes artiklis ka zig-zag teooriat. Käesolevas töös on laiendatud kõrgemat järku Haari lainikute meetod Timoshenko tala vabavõnkumiste analüüsiks. Töös on kasutatud jäik-jäik ja jäik-vaba (vaba toetus) rajatingimusi. Analüüsitud on erivate ristlõike muutumise koefitsentidele vastavaid lahendusi. Kõrgemat järku Haari lainikute meetod osutus täpseks ja kiireks nii 4. kui 6. järku koonduvuse korral (koonduvuse järk on määratud meetodi parameetriga). Saadud tulemused on üldistatavad laiema plaatide/koorikute vabavõnkumisi käsitlevate ülesannete klassi jaoks, kattes ka juhtusid, kus analüütiline lahend puudub. Saadud tulemused on kooskõlas laiemalt kasutatava Haari lainikute meetodi ja Ritzi meetodi abil saadud tulemustega. Kõrgemat järku Haari lainikute meetodit võib vaadelda kui Haari lainikute meetodi edasiarendust, mis tagab kõrgemat järku koonduvuskiiruse ja väiksema absoluutse vea.