

SEARCH FOR THE PARETO POINT BASED ON THE MAXIMIN PRINCIPLE OF IMPROVEMENT RATES OF OBJECTIVE FUNCTIONS

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The aim of this paper is to find the Pareto point for the numerical value of objective functions $F_1(x), \dots, F_n(x)$ with the argument x belonging to a certain Hilbert space. Starting from a fixed point x_0 , the optimization trajectory is chosen in such a way that on every step the improvement rate of the value of the slowest improvement of the objective function be possibly great. It is proved that on every step one has to choose the non-negative coefficients $\alpha_1, \dots, \alpha_n$ with the constraint $\sum_{i=1}^n \alpha_i = 1$, so that the norm of the gradient of the function $\sum_{i=1}^n \alpha_i F_i(x)$ is minimal, and then to make the next step in the direction of the chosen gradient.

Introduction

As known, optimization means finding such a point in some space of parameters where the objective function attains the best value. In the literature written in Estonian, the monograph [1] could give a good overview of the conventional optimization theory. Unlike common optimization, poly-optimization deals with more than one objective function. Although the space of parameters is the same for all these functions, as a rule, the best points of objective functions are different. In this case, a certain compromise point must be found in the space of parameters.

By setting the problem, two principally different cases should be distinguished. In the first case, the same stakeholder applies for the best values of all objective functions and thus finding the compromise point is an internal business. In the second case, different stakeholders stand behind different objective functions, but the point in the parameter space can be chosen based on the consensus only. If the consensus has not been reached, the problem tends to shift to the field of game theory that we shall not consider here.

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In both cases, the only solution could be some Pareto point of the parameters space. The point x is called the Pareto point or Pareto optimum when no point y can be found where the one value at least of one objective function would be better and that for the rest not worse than at the point x . This point y would be better than x in any sense and the latter would be of no interest any more. The set of all Pareto points is called the Pareto region. As a rule, it includes also the optimum of each single objective function.

Kotarski [2] gives a good summary about the present state of the search (or selection) theory of the Pareto optimum. This work pays a special attention to the results of Georgian mathematician Salukwadze [3–4]. However, these solutions are suitable for the first above-mentioned case where, based on a single man decision, the point can be freely shifted in the parameters space. For the consensus, shifting from the existing point can be in question only in such a way that the value of no objective function will deteriorate. This article is devoted first of all to this problem.

Our results reflect the studies initiated within the Baltic–German–Portuguese co-operation project within the Copernicus program in 1993–1996 [5–6].

In numerous cases, including the one in this project [5, 6], the problem set-up is even more general, so that the objective functions are given partially in the implicit form. Namely, the explicit functions $H_1(z), \dots, H_n(z)$, $z \in Z$ have been given, but Z is a space with a larger dimension than the actual domain of the functions. The dimension is namely constrained with the system of equations $W(z) = 0$ where the number of equations is less than the dimension of space Z . These equations determine the surface $S \subset Z$, the dimension of which is less than that of the space Z by the number of equations. The surface S is the actual domain of the objective functions.

In this case a subspace consisting of the components of the space Z is chosen with its dimension coinciding with the dimension of the space S . The vector x with these coordinates of the point z , which belongs to the said subspace, is called a vector of independent coordinates. The equations W give thus the implicit function φ , which determines the whole point z according to the vector x of independent coordinates of the point z in such a way that $W(\varphi(x)) = 0$. The objective functions $H_i(z)$ will change now into the implicit functions of the vector x : $F_i(x) = H_i(\varphi(x))$.

In addition, the boundary conditions $z_{\min} \leq z \leq z_{\max}$ may be set to the variables z where the inequalities between vectors mean inequalities for all the components at the same time. Then it means for the vectors x inequalities $z_{\min} \leq \varphi(x) \leq z_{\max}$. In the space of independent coordinates as many pairs of constraints appear then as large is the dimension of the space Z , while the constraints corresponding to the independent coordinates are planar, and the constraints relative to dependent coordinates are in general curved.

Since the distribution of coordinates into dependent and independent ones is in general free, when fixing at a certain point of the surface S , we shall consider first of all these coordinates independent relative to which the point

is on the boundary. When moving along this surface S to some other point located on the boundary of some hitherto dependent coordinate, we shall state this coordinate to be independent now and consider dependent any other independent coordinate relative to which the new point is not on the boundary. Therefore, we shall ensure that we have always-direct contact with planar boundaries.

Let us name herewith an extremely essential and efficient application that has been implemented in practice already, and that, as a matter of fact, turned our attention to the necessity of the following theory and led to its elaboration. Namely in the complex optimal control theory of big power systems (PS) and interconnected power systems (IPS), this approach has been implemented most widely and efficiently for solving hierarchic sub-problems of optimization within the framework of the so-called Generalized Reduced Gradient Method (GRGM) where, in addition to the implicit functions theory, a rational choice of a basis is used (i.e., the composition of the components of vectors of dependent and independent parameters) for speeding up the finding of an optimal solution [9–11].

In the Baltic–German–Portuguese joint project within the framework of the Copernicus program, a basis was laid for the generalization and development of GRGM providing the Pareto optimal correction of so-called trans-national equilibriums (determined by a certain free market mechanism) of IPS (first of all the Baltic IPS with its interstate dispatching center DC BALTIJA located in Riga) in behalf of all the partners under their certain consensus conditions [6–8].

Reduction of Poly-Optimization to Mono-Optimization

Let us have a certain finite number of objective functions F_1, \dots, F_n defined in the same space of parameters. Let us assume that it is the Hilbert space. Let the objective for each F_i be to find an x , such that the value of $F_i(x)$ would be as great as possible (in case the objective for some function is the diminution of its value, some mathematical transformation would enable to get a certain new function, the increase of which is equal to the decrease of the initial function). Let us assume also that the metrics of the values of various functions F_i is unified in a certain natural way so that the changing of their values can be mutually compared (for example, if the values are profits of various parties involved, the statement will be valid in case the currency exchange rates are fixed).

It is known (e.g., the proof can be found in [1]) that the point x is the Pareto optimum of the given functions just in case if and only if there exist numerical coefficients $\alpha_1, \dots, \alpha_n, \alpha_i \geq 0$ for each i and $\sum_{i=1}^n \alpha_i = 1$, so that x is the optimum

of the function $\sum_{i=1}^n \alpha_i F_i$. Thus the choice of the Pareto point is reduced to the selection of the numerical coefficients $\alpha_i, 1 \leq i \leq n$.

It is favorable for each specific F_i that its coefficient would be possibly great. If for example $\alpha_i = 1$ and the rest of coefficients are zeros, we get the optimum point of F_i . The transition from one Pareto point to another means changing of the coefficients α_i . As a rule, these objective functions whose coefficient increases win, and those whose coefficient decreases loose. Thus when one has reached some Pareto point already, one cannot move to another based on the consensus. It is easy to conclude from here that when the initial point from where we started optimization is close to some point in the Pareto region, only the Pareto points close to the initial point will be taken into account.

Considering this, the value of some objective function is always worse at the distant Pareto points than at those close to it as it inevitably is when moving from one point to another in the Pareto region and therefore evidently worse than at the initial point and that would make moving there based on the consensus impossible. The intuition tells that when approaching the Pareto region as orthogonally as the position of the Pareto region relative to the initial point allows, the direction of approach in respect to the gradients of all objective functions is acutely angled, and thus the values of all objective functions will be greater. Adding some component parallel to the Pareto region to the direction of movement, we may impair some objective function. Namely, we shall try to shift the point in the domain of definition of objective functions in such a way that the improvement rate of the most slowly improving objective function would be as high as possible. We call it the maximin principle of the improvement rate of objective functions.

Thus, a question arises how to choose the coefficients so that the optimum of the function $\sum_{i=1}^n \alpha_i F_i$ would be close to the initial point. The constraint

$\sum_{i=1}^n \alpha_i = 1$ creates a situation where only the proportion of the coefficients α_i is significant. Since the gradient of the function at the optimum point is a zero vector and close to the optimum point a vector with a small norm, the optimum of the function $\sum_{i=1}^n \alpha_i F_i$ is evidently close to the initial point if the choice of the coefficients α_i is such that the gradient norm of $\sum_{i=1}^n \alpha_i F_i$ at the initial point is small.

Selection of the Direction for the Next Optimization Step

Fixing some point in the space of parameters, we must choose the direction of the next optimization step. By mono-optimization the gradient of objective function determines the direction, because in this direction the objective function improves most rapidly. According to the considerations given in the previous subsection, the direction of the gradient of the function $\sum_{i=1}^n \alpha_i F_i$ must be chosen for the direction while the coefficients α_i must be chosen in such a way that the norm of this gradient would be minimal.

Denoting the gradient vector of each objective function F_i by F_i' , we can see that first of all the quantity $\left\| \sum_{i=1}^n \alpha_i F_i' \right\|$ must be minimized with the selection

of the coefficients α_i under the conditions $\alpha_i \geq 0$ for each i and $\sum_{i=1}^n \alpha_i = 1$.

The gradient vectors F_i' have naturally been computed at the point where we were before the next step. This is minimization of the value of a quadratic form, and this will not create any principal problem.

In case this point lies on a certain boundary (as known, we can assume that this is a flat boundary), and for the selected coefficients α_i the vector $\sum_{i=1}^n \alpha_i F_i'$ is directed outward the allowed domain, then assuming the continuity of the maximin of improvement rates of objective functions, we can see that the best among the allowed directions must lie just on the boundary. Then this boundary must be considered in the part of the space of independent coordinates at this step and the coefficients α_i calculated to take the step in this space of a smaller dimension. For this purpose, the similar norm minimization must be repeated, only each F_i' will be replaced by its projection onto this boundary.

If there are more than one boundary where this point lies, and free minimization of the norm gives a direction outward the feasible domain, among the boundaries passing that point the number of such boundaries must be chosen so that when minimizing the norm by the above mentioned method at the intersection of these boundaries as in a separate subspace, we can get the direction, which is an allowed direction also for the rest of boundaries where the point lies.

Having fixed the point under the conditions named in the space of coefficients α_i that minimizes the quantity $\left\| \sum_{i=1}^n \alpha_i F_i' \right\|$, we shall separate those that we name 'fixed to zero' from among the coefficients α_i . The coefficient α_j is called 'fixed to zero' when the following conditions are met

a) $\alpha_j = 0$;

b) Abandoning the constraint $\alpha_j = 0$ for this certain j , in the space of the coefficients α_i some shift can be made, and in this case $\left\| \sum_{i=1}^n \alpha_i F'_i \right\|$ will still decrease.

Let us split the indices $1 \leq i \leq n$ into two groups, I_1 and I_2 . The indices i where α_i has not been fixed to zero belong to the group I_1 , the rest to the group I_2 . At that I_2 may prove to be empty.

Improvement Rate of Objective Functions in the Preference Direction

The rate of change of a function in some direction is the scalar product of the gradient of this function and the unit vector of that direction. Let the direction be $\sum_{i=1}^n \alpha_i F'_i$ where the coefficients α_i have been obtained in the way shown in the previous section. Let us call this direction a preference direction.

Let $k \in I_1$ be such that $\alpha_k > 0$. Such an index can certainly be found. In the case where k is not the only element of I_1 , we shall choose an arbitrary $j \in I_1$, $j \neq k$ and compare the rate of change of the objective functions in the preference direction.

Since the given choice of coefficients α_i minimizes the quantity $\left\| \sum_{i=1}^n \alpha_i F'_i \right\|^2$, when changing it in such a way that α_k will be replaced by the difference $\alpha_k - \varepsilon$ and α_j by the sum $\alpha_j + \varepsilon$ (the sign of ε being arbitrary), we obtain instead of $\left\| \sum_{i=1}^n \alpha_i F'_i \right\|^2$ the expression

$\left\| \sum_{i=1}^n \alpha_i F'_i \right\|^2 + 2\varepsilon(F'_j - F'_k, \sum_{i=1}^n \alpha_i F'_i) + \varepsilon^2 \|F'_j - F'_k\|^2$, which attains the minimum for $\varepsilon = 0$, because $j \in I_1$ allows to exclude the restriction $\alpha_i \geq 0$, thereby without any change of the minimum point. Therefore $(F'_j - F'_k, \sum_{i=1}^n \alpha_i F'_i) = 0$, i.e., $(F'_j, \sum_{i=1}^n \alpha_i F'_i) = (F'_k, \sum_{i=1}^n \alpha_i F'_i)$ must be valid.

Thus the scalar product of F'_j and F'_k by the vector $\sum_{i=1}^n \alpha_i F'_i$, and also by the unit vector in the same direction are equal, i.e., F_j and F_k change equally in the preference direction.

As for $i \in I_2$ $\alpha_i = 0$, then

$$\left\| \sum_{i=1}^n \alpha_i F_i \right\|^2 = \left(\sum_{j \in I_1} \alpha_j F_j', \sum_{i=1}^n \alpha_i F_i' \right) = \sum_{j \in I_1} \alpha_j (F_j', \sum_{i=1}^n \alpha_i F_i').$$

As for a given $k \in I_1$ $(F_j', \sum_{i=1}^n \alpha_i F_i') = (F_k', \sum_{i=1}^n \alpha_i F_i')$ is also valid for each $j \in I_1$, we can present the final form of the considered expression as follows

$$\sum_{j \in I_1} \alpha_j (F_k', \sum_{i=1}^n \alpha_i F_i') = (F_k', \sum_{i=1}^n \alpha_i F_i') \sum_{j \in I_1} \alpha_j = (F_k', \sum_{i=1}^n \alpha_i F_i'), \text{ since } \sum_{j \in I_1} \alpha_j = 1.$$

Thus $(F_k', \sum_{i=1}^n \alpha_i F_i') = \left\| \sum_{i=1}^n \alpha_i F_i \right\|^2$ and for each $j \in I_1$,

$$(F_j', \sum_{i=1}^n \alpha_i F_i') = \left\| \sum_{i=1}^n \alpha_i F_i \right\|^2 \geq 0.$$

The last inequality is strict when the direction vector of the preference direction is not a zero vector, i.e., when this point remains outside the Pareto region.

Let us formulate the result as follows.

Theorem 1. In the preference direction all the objective functions $F_j, j \in I_1$ will improve and do it with equal rate.

Let us choose now $j \in I_2$ and this time $\varepsilon > 0$. If now to replace α_k with the difference $\alpha_k - \varepsilon$ and α_j with the sum $\alpha_j + \varepsilon$, the square of the norm gradient of the function $\sum_{i=1}^n \alpha_i F_i$ can only increase, because the change remained within the allowed range of restrictions. Thus, the linear or principal part of the change $2\varepsilon(F_j' - F_k', \sum_{i=1}^n \alpha_i F_i')$ must be non-negative. It means that

$(F_j', \sum_{i=1}^n \alpha_i F_i') \geq (F_k', \sum_{i=1}^n \alpha_i F_i')$, which shows that F_j will improve in the preference direction at least as fast as an arbitrary $F_k, k \in I_1$.

Moreover, since $j \in I_2$, a shift is possible in the space of the coefficients α_i by eliminating the restriction $\alpha_j > 0$ so that the square of the norm of the gradient $\sum_{i=1}^n \alpha_i F_i$ will decrease. In this case α_j will be replaced by the difference $\alpha_j - \varepsilon, \varepsilon > 0$, while for each $i \neq j, \alpha_i, \alpha_i$ will be replaced by the sum $\alpha_i + \varepsilon_i$, whereby $\sum_{i \neq j} \varepsilon_i = \varepsilon$ and for these $i \neq j$ where $\alpha_i = 0, \varepsilon_i$ must be certainly non-negative. Then the principal changing part of the square of the

norm of the above gradient is $2(\sum_{i \neq j} \varepsilon_i F'_i - \varepsilon F'_j, \sum_{i=1}^n \alpha_i F'_i)$, which must be negative. Thus, $\sum_{i \neq j} \varepsilon_i (F'_i, \sum_{h=1}^n \alpha_h F'_h) < \varepsilon (F'_j, \sum_{h=1}^n \alpha_h F'_h)$.

If for some $i \neq j$, $\varepsilon_i < 0$ (if any found), then $\alpha_i > 0$ and thus $i \in I_1$. In this case $-\varepsilon_i (F'_i, \sum_{h=1}^n \alpha_h F'_h) \leq -\varepsilon_i (F'_j, \sum_{h=1}^n \alpha_h F'_h)$, because according to the above, this inequality between the scalar products is valid and the coefficient $-\varepsilon$ is positive. By adding the sides of this non-strict inequality to the corresponding sides of the preceding strict inequality, the strict inequality will remain valid. Applying the same procedure for each $i \neq j$, where $\varepsilon_i < 0$, in turn, we shall get the inequality $\sum_{\substack{i \neq j \\ \varepsilon_i \geq 0}} \varepsilon_i (F'_i, \sum_{h=1}^n \alpha_h F'_h) < (\sum_{\substack{i \neq j \\ \varepsilon_i \geq 0}} \varepsilon_i) (F'_j, \sum_{h=1}^n \alpha_h F'_h)$. As in case of

the above mentioned $k \in I_1$ $(F'_k, \sum_{h=1}^n \alpha_h F'_h) \leq (F'_i, \sum_{h=1}^n \alpha_h F'_h)$ for each i , then replacing all the scalar products on the left-hand side of the strict inequality with the scalar product $(F'_k, \sum_{h=1}^n \alpha_h F'_h)$, the strict inequality will remain valid. Thus $(\sum_{\substack{i \neq j \\ \varepsilon_i \geq 0}} \varepsilon_i) (F'_k, \sum_{h=1}^n \alpha_h F'_h) < (\sum_{\substack{i \neq j \\ \varepsilon_i \geq 0}} \varepsilon_i) (F'_j, \sum_{h=1}^n \alpha_h F'_h)$ will be valid. Let us note that since $\varepsilon = \sum_{i \neq j} \varepsilon_i$, then $\sum_{\substack{i \neq j \\ \varepsilon_i \geq 0}} \varepsilon_i \geq \varepsilon > 0$.

From here $(F'_k, \sum_{h=1}^n \alpha_h F'_h) < (F'_j, \sum_{h=1}^n \alpha_h F'_h)$. The result could be formulated as follows.

Theorem 2. In the preference direction any $F_j, j \in I_2$ will improve faster than an arbitrary $F_i, i \in I_1$.

Change of the Minimum Improvement Rates of Objective Functions by the Deviation from the Preference Direction

Let us examine what happens to the improvement rates when we do not start to move in the preference direction in order to take an optimization step, but somewhat in another direction. Based on the maximin principle, we are first of all interested in the fact how the improvement rate of these objective functions will change, which improve with the slowest rate in the preference direction, i.e., the objective functions $F_i, i \in I_1$.

Having fixed a certain direction, we should consider the scalar product of the unit vector of this direction and gradient vector $F'_i, i \in I_1$. It would be the

same as to project this unit vector onto the vector F'_i and multiply this projection by the length of F'_i .

Let us consider now the subspace formed by the gradient vectors $F'_i, i \in I_1$. Projecting a unit vector of any direction onto a certain vector F'_i of them, we can project it first of all onto the mentioned subspace and then project this projection onto the vector F'_i . The result would be the same. As by projecting onto a subspace the length of the vector can only decrease, then, consequently, we would get a better result if we elongate the projection of this unit vector onto the subspace to the unit length before projecting it onto the vector F'_i . This is equivalent to the situation where we would have replaced the considered direction with its projection onto the mentioned subspace.

Thus it is reasonable to consider only these directions, which are located in the subspace formed by the vectors $F'_i, i \in I_1$, so as by the substitution of any other direction with its projection onto this subspace the improvement rates of all $F'_i, i \in I_1$, could only increase.

Let the direction in the mentioned subspace be given in the form $\sum_{i \in I_1} \beta_i F'_i$.

Since only the proportion of the coefficients β_i is essential, we can assume

$$\sum_{i \in I_1} \beta_i = 1, \text{ i.e., } \beta_i = \alpha_i + \varepsilon_i \text{ where } \sum_{i \in I_1} \varepsilon_i = 0. \text{ In this case, } \left\| \sum_{i \in I_1} \beta_i F'_i \right\| \geq \left\| \sum_{i \in I_1} \alpha_i F'_i \right\|$$

and $\left\| \sum_{i \in I_1} \beta_i F'_i \right\|^2$ can be expressed in the form

$$\left\| \sum_{i \in I_1} \alpha_i F'_i \right\|^2 + 2 \left(\sum_{i \in I_1} \varepsilon_i F'_i, \sum_{i \in I_1} \alpha_i F'_i \right) + \left\| \sum_{i \in I_1} \varepsilon_i F'_i \right\|^2. \text{ Assuming that } \sum_{i \in I_1} \varepsilon_i = 0, \text{ this}$$

expression must be greater than $\left\| \sum_{i \in I_1} \alpha_i F'_i \right\|^2$ or equal to it (if $\varepsilon_i = 0$ for each i).

Based on the same assumption $\left(\sum_{i \in I_1} \varepsilon_i F'_i, \sum_{i \in I_1} \alpha_i F'_i \right)$ must equal zero, i. e.

the coefficients $\left(F'_i, \sum_{h \in I_1} \alpha_h F'_h \right)$ of all ε_i in this linear form are equal since otherwise it could be changed to become negative, changing, if necessary, the signs of all ε_i to negative, and thereafter by the proportional approximation of all ε_i to zero make $\left\| \sum_{i \in I_1} \varepsilon_i F'_i \right\|^2$ insignificant in respect of the considered scalar product.

The requirement $\left(\sum_{i \in I_1} \varepsilon_i F'_i \right) = 0$ with the condition $\sum_{i \in I_1} \varepsilon_i = 0$ means that all multipliers of ε_i in this linear form, i.e. the scalar products $\left(F'_i, \sum_{h \in I_1} \alpha_h F'_h \right)$ should be equal, since otherwise we could change this linear

form to become different from zero, choosing for the values of ε_j and ε_k , having different multipliers, the values of ε and $-\varepsilon$ respectively and taking the rest of ε_i equal to zero.

$$\text{However, } \left(\sum_{i \in I_1} \varepsilon_i F'_i, \sum_{i \in I_1} \alpha_i F'_i \right) = \sum_{i \in I_1} \alpha_i (F'_i, \sum_{h \in I_1} \varepsilon_h F'_h).$$

Thus, if $\sum_{i \in I_1} \varepsilon_i = 0$, $(F'_i, \sum_{h \in I_1} \varepsilon_h F'_h)$ cannot be strictly positive for each $i \in I_1$.

But if for some $i \in I_1$ $(F'_i, \sum_{h \in I_1} \varepsilon_h F'_h) = (F'_i, \sum_{h \in I_1} \beta_h F'_h) - (F'_i, \sum_{h \in I_1} \alpha_h F'_h) \leq 0$, then $(F'_i, \sum_{h \in I_1} \beta_h F'_h) \leq (F'_i, \sum_{h \in I_1} \alpha_h F'_h)$. (*)

In case this point is not the Pareto point, then based on the *Theorem 1* the right-hand side of the last inequality is strictly positive.

If the vectors $\sum_{h \in I_1} \beta_h F'_h$ and $\sum_{h \in I_1} \alpha_h F'_h$ are replaced with the unit vectors of

the same direction in the last inequality, then in case if $\left\| \sum_{h \in I_1} \beta_h F'_h \right\| > \left\| \sum_{h \in I_1} \alpha_h F'_h \right\|$, the inequality (*) will become strict. It means that for i in this inequality, F_i will increase slower in the direction $\sum_{h \in I_1} \beta_h F'_h$ than in the preference direction.

Thus, we have proved the following result:

Theorem 3. At any deviation from the preference direction there exists at least one F'_i , $i \in I_1$, the improvement rate of which decreases against the preference direction.

Thus based on the maximin principle of improvement rates the direction $\sum_{i=1}^n \alpha_i F'_i$ is the most preferred direction.

Implementation

The given results would be implemented according to the following scheme.

Let F_1, \dots, F_n be the given objective functions. Let us assume that the initial point in the parameter space is x_0 , which is not the Pareto optimum. Let us find the coefficients $\alpha_1, \dots, \alpha_n$ on the condition that for each i $\alpha_i \geq 0$ and $\sum_{i=1}^n \alpha_i = 1$

while under these conditions at the point x_0 the value of $\left\| \sum_{i=1}^n \alpha_i F'_i \right\|$ would be minimal. Then we shall take the optimization step in such a way as for the conventional mono-optimization when the objective function would be $\sum_{i=1}^n \alpha_i F_i$. As shown above, the step will be taken in the direction that proceeds from the maximin principle of the improvement rate of objective functions. When reaching the point x_1 with this step, we shall take it for an initial point

and repeat the similar procedure. The process will continue until we have reached the point where in case of choosing the appropriate quantities of α_i , the value $\left\| \sum_{i=1}^n \alpha_i F_i \right\|$ is practically zero.

As mentioned in the introduction, the given conceptions have been used for coordinating the performance of power systems of three Baltic countries within the Copernicus programme in 1993–1996 with an objective to maximize energy conservation in the power systems of each state. In the calculations the respective development of GRGM within the programme set KORONA at the disposal of SC BALTIJA was used [5,6].

To the present time this method has been developed further in the game theory approach for the control of international IPS considering both a certain consensus of partner coalitions and disagreements of these coalitions, especially in case of major disturbances [7, 8]. In the case of disagreements between partners instead of the methods given above the methods of game theory must be used [12].

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