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MATHEMATICS

On endomorphisms of groups of order 36

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Abstract. There exist exactly 14 non-isomorphic groups of order 36. In this paper we will prove that three of them are not determined by their endomorphism semigroups in the class of all groups. All groups that have an endomorphism semigroup isomorphic to the endomorphism semigroup of a group of order 36 are described.

Key words: group, semigroup, endomorphism semigroup.

1. INTRODUCTION

It is well known that all endomorphisms of an Abelian group form a ring and many of their properties can be characterized by this ring. An excellent overview of the present situation in the theory of endomorphism rings of Abelian groups is given by Krylov et al. [3]. All endomorphisms of an arbitrary group form only a semigroup. The theory of endomorphism semigroups of groups is quite modestly developed. In a number of our papers we have made efforts to describe some classes of groups that are determined by their endomorphism semigroups in the class of all groups. Let G be a group. If for each group H such that the semigroups $\mathrm{End}(G)$ and $\mathrm{End}(H)$ are isomorphic implies an isomorphism between G and H, then we say that the group G is determined by its endomorphism semigroup in the class of all groups. Examples of such groups are finite Abelian groups ([4], Theorem 4.2), generalized quaternion groups ([5], Corollary 1), torsion-free divisible Abelian groups ([6], Theorem 1), etc.

We know a complete answer to this problem for finite groups of order less than 36. The alternating group A_4 (also called the tetrahedral group) and the binary tetrahedral group $B = \langle a, b \mid b^3 = 1,\ aba = bab\rangle$ are the only groups of order less than 32 that are not determined by their endomorphism semigroups in the class of all groups [12]. These two groups are non-isomorphic, but their endomorphism semigroups are isomorphic. We have proved that each group of order 32 is determined by its endomorphism semigroup in the class of all groups: it has partly been made in published papers [13,14] and partly in papers to be published. The groups of orders 33 and 35 are cyclic, and, therefore, are determined by their endomorphism semigroups in the class of all groups ([4], Theorem 4.2). There exist two non-isomorphic groups of order 34: the cyclic group of order 34 and the dihedral group of order 34. Both are determined by their endomorphism semigroups in the class of all groups ([4], Theorem 4.2 and [10], Section 5).

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In this paper, we present a solution to the problem whose groups of order 36 are determined by their endomorphism semigroups. The group theoretical computer algebra system GAP contains the 'Small Groups Library', which provides access to descriptions of all groups of order 36 ([17]). There exist exactly 14 non-isomorphic groups of order 36. Throughout this paper, let us denote these groups by $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{14}$, respectively. The last three groups among them are

$$\mathcal{G}_{12} = \langle a, b, c \mid c^9 = b^2 = a^2 = 1,\ ab = ba,\ c^{-1}ac = b, c^{-1}bc = ab \rangle,$$
$$\mathcal{G}_{13} = C_3 \times A_4,$$
$$\mathcal{G}_{14} = \langle a, b, c \mid c^4 = a^3 = b^3 = 1,\ ab = ba,\ c^{-1}ac = b,\ c^{-1}bc = a^{-1} \rangle,$$

where C_3 is the cyclic group of order 3 and A_4 is the alternating group of order 12 (the tetrahedral group). In this paper, the following theorem is proved:

Theorem 1.1 (**Main theorem).** *The following statements hold for a group* G:

(1) *if the endomorphism semigroups of* G *and* $\mathcal{G}_i,\ i \in \{1, 2, \ldots, 11\}$ *are isomorphic, then* G *and* $\mathcal{G}_i$ *are isomorphic*;

(2) *the endomorphism semigroups of* G *and* $\mathcal{G}_{12}$ *are isomorphic if and only if* $G = \mathcal{G}_{12}$ *or*

$$G = \langle a, b, c \mid c^9 = a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1},\ c^{-1}bc = a, c^{-1}ac = ab \rangle;$$

(3) *the endomorphism semigroups of* G *and* $\mathcal{G}_{13}$ *are isomorphic if and only if* $G = \mathcal{G}_{13}$ *or* $G = C_3 \times B$, *where* $B = \langle a, b \mid b^3 = 1,\ aba = bab \rangle$ *is the binary tetrahedral group*;

(4) *the endomorphism semigroups of* G *and* $\mathcal{G}_{14}$ *are isomorphic if and only if* $G = \mathcal{G}_{14}$ *or*

$$G = \langle a, b, c, d \mid c^4 = a^3 = b^3 = d^3 = 1,\ ab = bad,\ c^{-1}ac = b,$$
$$c^{-1}bc = a^{-1},\ cd = dc,\ ad = da,\ bd = db \rangle.$$

We shall use the following notations:

G – a group;
$Z(G)$ – the centre of a group G;
G' – the derived subgroup of G;
$[a, b] = a^{-1}b^{-1}ab$ $(a, b \in G)$;
$C_G(a)$ – the centralizer of a in G;
$\mathrm{End}(G)$ – the endomorphism semigroup of G;
C_k – the cyclic group of order k;
A_4 – the alternating group of order 12 (the tetrahedral group);
$D_n = \langle a, b \mid b^2 = a^k = 1,\ b^{-1}ab = a^{-1} \rangle$ – the dihedral group of order $n = 2k$;
$B = \langle a, b \mid b^3 = 1,\ aba = bab \rangle$ – the binary tetrahedral group;
$\mathbb{Z}_k$ – the residue class ring $\mathbb{Z}/k\mathbb{Z}$;
$\mathbb{Z}_k[x]$ – the polynomial ring over $\mathbb{Z}_k$;
$\langle K, \ldots, g, \ldots \rangle$ – the subgroup generated by subsets $K, \ldots$ and elements $g, \ldots$;
$\widehat{g}$ – the inner automorphism of G, generated by an element $g \in G$;
$I(G)$ – the set of all idempotents of $\mathrm{End}(G)$;
$K(x) = \{z \in \mathrm{End}(G) \mid zx = xz = z\}$;
$J(x) = \{z \in \mathrm{End}(G) \mid zx = xz = 0\}$;
$V(x) = \{z \in \mathrm{Aut}(G) \mid zx = x\}$;
$D(x) = \{z \in \mathrm{Aut}(G) \mid zx = xz = x\}$;
$H(x) = \{z \in \mathrm{End}(G) \mid xz = z,\ zx = 0\}$;
$[x] = \{z \in I(G) \mid xz = z,\ zx = x\},\ x \in I(G)$;
$G = A \leftthreetimes B$ – G is a semidirect product of an invariant subgroup A and a subgroup B.

The sets $K(x), V(x), D(x)$, and $J(x)$ are subsemigroups of $\mathrm{End}(G)$; however, $V(x)$ and $D(x)$ are subgroups of $\mathrm{Aut}(G)$. We shall write the mapping right from the element on which it acts.

2. PRELIMINARIES

For the convenience of the reader, let us recall some known facts that will be used in the proofs of our main results.

Lemma 2.1. *If* $x \in I(G)$, *then* $G = \mathrm{Ker}\,x \leftthreetimes \mathrm{Im}\,x$ *and* $\mathrm{Im}\,x = \{\, g \in G \mid gx = g \,\}$.

Lemma 2.2. *If* $x \in I(G)$, *then* $[x] = \{y \in I(G) \mid \mathrm{Ker}\,x = \mathrm{Ker}\,y\}$.

Lemma 2.3. *If* $x \in I(G)$, *then*

$$K(x) = \{\, y \in \mathrm{End}(G) \mid (\mathrm{Im}\,x)y \subset \mathrm{Im}\,x,\ (\mathrm{Ker}\,x)y = \langle 1 \rangle \,\}$$

and $K(x)$ *is a subsemigroup with the unity* x *of* $\mathrm{End}(G)$, *which is canonically isomorphic to* $\mathrm{End}(\mathrm{Im}\,x)$. *Under this isomorphism element* y *of* $K(x)$ *corresponds to its restriction onto the subgroup* $\mathrm{Im}\,x$ *of* G.

Lemma 2.4. *If* $x \in I(G)$, *then*

$$H(x) = \{\, y \in \mathrm{End}(G) \mid (\mathrm{Im}\,x)y \subset \mathrm{Ker}\,x,\ (\mathrm{Ker}\,x)y = \langle 1 \rangle \,\}.$$

Lemma 2.5. *If* $x \in I(G)$, *then*

$$J(x) = \{z \in \mathrm{End}(G) \mid (\mathrm{Im}\,x)z = \langle 1 \rangle,\ (\mathrm{Ker}\,x)z \subset \mathrm{Ker}\,x\}.$$

Lemma 2.6. *If* $x \in I(G)$, *then*

$$D(x) = \{\, y \in \mathrm{Aut}(G) \mid y|_{\mathrm{Im}\,x} = 1|_{\mathrm{Im}\,x},\ (\mathrm{Ker}\,x)y \subset \mathrm{Ker}\,x\}.$$

Lemma 2.7. *If* $z \in \mathrm{End}(G)$ *and* $\mathrm{Im}\,z$ *is Abelian*, *then* $\widehat{g} \in V(z)$ *for each* $g \in G$.

We omit the proofs of these lemmas because these are straightforward corollaries from the definitions.

Lemma 2.8 ([4], Theorem 4.2)**.** *Every finite Abelian group is determined by its endomorphism semigroup in the class of all groups.*

Lemma 2.9 ([4], Theorem 1.13)**.** *If* G *and* H *are groups such that their endomorphism semigroups are isomorphic and* G *splits into a direct product* $G = G_1 \times G_2$ *of its subgroups* G_1 *and* G_2, *then* H *splits into a direct product* $H = H_1 \times H_2$ *of its subgroups* H_1 *and* H_2 *such that* $\mathrm{End}(G_1) \cong \mathrm{End}(H_1)$ *and* $\mathrm{End}(G_2) \cong \mathrm{End}(H_2)$.

From here follow Lemmas 2.10–2.13.

Lemma 2.10. *If groups* G_1 *and* G_2 *are determined by their endomorphism semigroups in the class of all groups*, *then so is their direct product* $G_1 \times G_2$.

Lemma 2.11 ([10], Section 5)**.** *The dihedral group* D_n *is determined by its endomorphism semigroup in the class of all groups.*

Lemma 2.12 ([9], Theorem, Lemmas 4.5–4.8)**.** *Let*

$$G = \langle a, b \mid a^{p^n} = b^v = 1,\ b^{-1}ab = a^r \rangle = \langle a \rangle \leftthreetimes \langle b \rangle = C_{p^n} \leftthreetimes C_v,$$

where p *is a prime*, $p > 2$, *and let* G^* *be another group such that the endomorphism semigroups of* G *and* G^* *are isomorphic. Assume that* x *is the projection of* G *onto its subgroup* $\langle b \rangle$ *and* x^* *corresponds to* x *under the isomorphism* $\mathrm{End}(G) \cong \mathrm{End}(G^*)$. *Then* G *and* G^* *are isomorphic and*

$$G^* = \langle c, d \mid c^{p^n} = d^v = 1,\ d^{-1}cd = c^{r^*} \rangle = \langle c \rangle \leftthreetimes \langle d \rangle = C_{p^n} \leftthreetimes C_v,$$

where $\mathrm{Im}\,x^* = \langle d \rangle$, $\mathrm{Ker}\,x^* = \langle c \rangle$, *and* $\langle r \rangle = \langle r^* \rangle$ *in the group of units of* $\mathbb{Z}_{p^n}$.

Lemma 2.13 ([2], Theorem 2.1 and Lemma 2.2)**.** *If G is a group such that $G/Z(G)$ is Abelian, then*

$$[g, hk] = [g, h] \cdot [g, k], \;\; [hk, g] = [h, g] \cdot [k, g],$$

$$(gh)^m = g^m h^m [h, g]^{m(m-1)/2},$$

$$[g, h]^m = [g^m, h] = [g, h^m]$$

for each $g, k, h \in G$ and positive integer m.

3. GROUPS OF ORDER 36

The group theoretical computer algebra system GAP provides access to descriptions of small order groups [17]. At present, the library of small order groups contains the groups of order at most 2000, except for order 1024 (423 164 062 groups).

There are 14 pairwise non-isomorphic groups of order 36. Following [17], they are

$\mathcal{G}_1 \cong C_{36}$; $\mathcal{G}_2 \cong C_{18} \times C_2$; $\mathcal{G}_3 \cong C_{12} \times C_3$; $\mathcal{G}_4 \cong C_6 \times C_6$;

$\mathcal{G}_5 = \langle a, b \mid a^4 = b^9 = 1,\ a^{-1}ba = b^{-1} \rangle \cong C_9 \rtimes C_4$;

$\mathcal{G}_6 = \langle a, b, d \mid a^4 = b^3 = d^3 = 1,\ a^{-1}dad = 1,\ b^{-1}a^{-1}ba = 1,\ d^{-1}b^{-1}db = 1 \rangle$;

$\mathcal{G}_7 = \langle a, b, c \mid a^2 = b^2 = c^9 = 1,\ (ac^{-1})^2 = 1,\ (ba)^2 = 1,\ c^{-1}bcb = 1 \rangle$;

$\mathcal{G}_8 = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^3 = 1,\ (ad)^2 = 1,\ (bc)^2 = 1,\ (ba)^2 = 1,$
$c^{-1}aca = 1,\ d^{-1}bdb = 1,\ d^{-1}c^{-1}dc = 1 \rangle$;

$\mathcal{G}_9 = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^3 = 1,\ (ad)^2 = 1,\ (ba)^2 = 1,$
$c^{-1}aca = 1,\ c^{-1}bcb = 1,\ d^{-1}bdb = 1,\ d^{-1}c^{-1}dc = 1 \rangle$;

$\mathcal{G}_{10} = \langle a, c, d \mid a^4 = c^3 = d^3 = 1,\ a^{-1}cac = 1,\ a^{-1}dad = 1,$
$d^{-1}c^{-1}dc = 1,\ (a^{-1}c)^2 a^{-2} = 1,\ c^{-1}da^{-1}dc^{-1}a = 1 \rangle$;

$\mathcal{G}_{11} = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^3 = 1,\ (ac)^2 = 1,\ (ad)^2 = 1,$
$(ba)^2 = 1,\ c^{-1}bcb = 1,\ d^{-1}bdb = 1,\ d^{-1}c^{-1}dc = 1 \rangle$;

$\mathcal{G}_{12} = \langle a, c \mid a^9 = c^2 = 1,\ (a^{-1}c)^2 \cdot a^2 c = 1 \rangle$;

$\mathcal{G}_{13} = \langle a, b, c \mid c^2 = b^3 = a^3 = 1,\ (ca)^3 = 1,\ (a^{-1}c)^3 = 1,$
$b^{-1}a^{-1}ba = 1,\ cb^{-1}cb = 1 \rangle$;

$\mathcal{G}_{14} = \langle a, c \mid a^4 = c^3 = 1,\ a^{-1}ca^2ca^{-1} = 1,\ a^{-1}caca^{-1}c^{-1}ac^{-1} = 1 \rangle$.

One of the main ideas to prove the main theorem is to use Lemmas 2.1–2.13. Therefore, we present some of the groups $\mathcal{G}_i$ $(1 \le i \le 14)$ in the form suitable for these lemmas.

(1) The group $\mathcal{G}_6$. It follows from the determining relations of $\mathcal{G}_6$:

$$ab = ba,\ bd = db;\ \ a^{-1}dad = 1 \Longrightarrow a^{-1}da = d^{-1},$$

and, therefore,

$$\mathcal{G}_6 = \langle b \rangle \times (\langle d \rangle \rtimes \langle a \rangle) \cong C_3 \times (C_3 \rtimes C_4). \tag{3.1}$$

(2) The group $\mathscr{G}_7$. It follows from the determining relations of $\mathscr{G}_7$:

$$(ba)^2 = 1 \Longrightarrow ab = ba,\ c^{-1}bcb = 1 \Longrightarrow bc = cb,$$

$$(ac^{-1})^2 = 1 \Longrightarrow ac^{-1}ac^{-1} = 1 \Longrightarrow ac^{-1}a = c \Longrightarrow aca = c^{-1} \Longrightarrow a^{-1}ca = c^{-1}.$$

Therefore,

$$\mathscr{G}_7 = \langle b\rangle \times (\langle c\rangle \leftthreetimes \langle a\rangle) \cong C_2 \times D_{18}. \tag{3.2}$$

(3) The group $\mathscr{G}_8$. Rewriting the determining relations of $\mathscr{G}_8$, we get

$$a^{-1}da = d^{-1},\ b^{-1}cb = c^{-1},\ ab = ba,\ ac = ca,\ bd = db,\ dc = cd.$$

Hence

$$\mathscr{G}_8 = (\langle d\rangle \leftthreetimes \langle a\rangle) \times (\langle c\rangle \leftthreetimes \langle b\rangle) \cong D_6 \times D_6. \tag{3.3}$$

(4) The group $\mathscr{G}_9$. It follows from the determining relations of $\mathscr{G}_9$:

$$dc = cd,\ db = bd,\ bc = cb,\ ac = ca,$$

$$(ad)^2 = 1 \Longrightarrow a^{-1}da = d^{-1} \Longrightarrow \langle a, d\rangle = \langle d\rangle \leftthreetimes \langle a\rangle \cong D_6,$$

$$\mathscr{G}_9 = \langle a, d\rangle \times \langle c\rangle \times \langle b\rangle \cong D_6 \times C_3 \times C_2 \cong D_6 \times C_6. \tag{3.4}$$

(5) The group $\mathscr{G}_{10}$. It follows from the determining relations of $\mathscr{G}_{10}$:

$$a^{-1}ca = c^{-1},\ a^{-1}da = d^{-1},\ cd = dc,$$

$$\begin{aligned}(a^{-1}c)^2a^{-2} &= a^{-1}ca^{-1}ca^{-2} = a^{-1}c \cdot a^{-1}ca \cdot a^{-3}\\ &= a^{-1}c \cdot c^{-1} \cdot a^{-3} = a^{-4} = 1,\end{aligned}$$

$$c^{-1}da^{-1}dc^{-1}a = c^{-1}d \cdot a^{-1}da \cdot a^{-1}c^{-1}a = c^{-1}d \cdot d^{-1} \cdot c = 1.$$

Hence

$$\langle c\rangle \leftthreetimes \langle a\rangle \cong \langle d\rangle \leftthreetimes \langle a\rangle \cong C_3 \leftthreetimes C_4$$

and

$$\begin{aligned}\mathscr{G}_{10} &= (\langle c\rangle \times \langle d\rangle) \leftthreetimes \langle a\rangle = \langle c\rangle \leftthreetimes (\langle d\rangle \leftthreetimes \langle a\rangle)\\ &= \langle d\rangle \leftthreetimes (\langle c\rangle \leftthreetimes \langle a\rangle) \cong (C_3 \times C_3) \leftthreetimes C_4.\end{aligned}$$

(6) The group $\mathscr{G}_{11}$. It follows from the determining relations of $\mathscr{G}_{11}$:

$$dc = cd,\ db = bd,\ bc = cb,\ ab = ba,$$

$$(ad)^2 = 1 \Longrightarrow a^{-1}da = d^{-1};\ \langle a, d\rangle = \langle d\rangle \leftthreetimes \langle a\rangle \cong C_3 \leftthreetimes C_2 \cong D_6,$$

$$(ac)^2 = 1 \Longrightarrow a^{-1}ca = c^{-1};\ \langle a, c\rangle = \langle c\rangle \leftthreetimes \langle a\rangle \cong C_3 \leftthreetimes C_2 \cong D_6,$$

$$\langle a, c, d\rangle = (\langle c\rangle \times \langle d\rangle) \leftthreetimes \langle a\rangle = \langle c\rangle \leftthreetimes (\langle d\rangle \leftthreetimes \langle a\rangle) = \langle d\rangle \leftthreetimes (\langle c\rangle \leftthreetimes \langle a\rangle),$$

$$\mathscr{G}_{11} = \langle b\rangle \times \langle a, c, d\rangle \cong C_2 \times ((C_3 \times C_3) \leftthreetimes C_2).$$

(7) The group $\mathscr{G}_{12}$. It follows from the determining relations of $\mathscr{G}_{12}$:

$$a^{-1}ca^{-1}ca^2c = 1 \Longrightarrow ca^{-1}ca^2 = ac \Longrightarrow ca^{-1}ca = aca^{-1},$$

$$a^{-1}ca^{-1}ca^2c = 1 \Longrightarrow ca^2c = aca,$$
$$c = a^{-1} \cdot aca \cdot a^{-1} = a^{-1} \cdot ca^2c \cdot a^{-1} = a^{-1}ca \cdot aca^{-1}$$
$$= (a^{-1}ca)^{-1} \cdot aca^{-1} \Longrightarrow a^{-1}ca \cdot c = aca^{-1} = c \cdot a^{-1}ca,$$
$$\langle c, a^{-1}ca\rangle = \langle c\rangle \times \langle a^{-1}ca\rangle \cong C_2 \times C_2,$$
$$a^{-1} \cdot a^{-1}ca \cdot a = a^{-2} \cdot ca^2 = a^{-2} \cdot acac = a^{-1}ca \cdot c,$$
$$\mathscr{G}_{12} = (\langle c\rangle \times \langle a^{-1}ca\rangle) \rightthreetimes \langle a\rangle \cong (C_2 \times C_2) \rightthreetimes C_9.$$

Denote the elements a, c, and $a^{-1}ca$ by c, a, and b, respectively. Then we get

$$\mathscr{G}_{12} = \langle a, b, c \mid c^9 = b^2 = a^2 = 1,\ ab = ba,\ c^{-1}ac = b, c^{-1}bc = ab\rangle$$
$$= (\langle a\rangle \times \langle b\rangle) \rightthreetimes \langle c\rangle \cong (C_2 \times C_2) \rightthreetimes C_9.$$

(8) The group $\mathscr{G}_{13}$. It follows from the determining relations of $\mathscr{G}_{13}$:

$$ab = ba,\ bc = cb,\ \langle a, b\rangle = \langle a\rangle \times \langle b\rangle \cong C_3 \times C_3,$$
$$(ca)^3 = 1 \Longrightarrow cacaca = 1 \Longrightarrow a^{-1}ca^{-1} = cac,$$
$$(a^{-1}c)^3 = a^{-1}ca^{-1} \cdot ca^{-1}c = cac \cdot ca^{-1}c = 1,$$
$$a^{-1}ca \cdot c = a^{-1} \cdot cac = a^{-1} \cdot a^{-1}ca^{-1} = aca^{-1},$$
$$c \cdot a^{-1}ca = ca^{-1}c \cdot a = (cac)^{-1} \cdot a = (a^{-1}ca^{-1})^{-1} \cdot a = aca \cdot a = aca^{-1},$$
$$\langle c\rangle \times \langle a^{-1}ca\rangle \cong C_2 \times C_2,$$
$$a^{-1} \cdot a^{-1}ca \cdot a = a^{-2}ca^2 = aca^{-1} = c \cdot a^{-1}ca,$$
$$\mathscr{G}_{13} = \langle b\rangle \times ((\langle c\rangle \times \langle a^{-1}ca\rangle) \rightthreetimes \langle a\rangle) \cong C_3 \times ((C_2 \times C_2) \rightthreetimes C_3).$$

Denote the elements $a^{-1}ca$, c, a, and b by b, a, c, and d, respectively. Then

$$\langle a, b, c\rangle = \langle a, b, c \mid c^3 = b^2 = a^2 = 1,\ ab = ba, c^{-1}ac = b,\ c^{-1}bc = ab\rangle = A_4,$$
$$\mathscr{G}_{13} = \langle d\rangle \times ((\langle a\rangle \times \langle b\rangle) \rightthreetimes \langle c\rangle) \cong C_3 \times A_4.$$

(9) The group $\mathscr{G}_{14}$. It follows from the determining relations of $\mathscr{G}_{14}$:

$$a^{-1}ca^2ca^{-1} = 1 \Longrightarrow ac^{-1} = a^{-1}ca^2 \Longrightarrow a^{-2}ca^2 = c^{-1},$$
$$a^{-1}caca^{-1}c^{-1}ac^{-1} = 1 \Longrightarrow a^{-1}ca = c \cdot a^{-1}ca \cdot c^{-1} \Longrightarrow$$
$$\Longrightarrow c \cdot a^{-1}ca = a^{-1}ca \cdot c,$$
$$\langle c, a^{-1}ca\rangle = \langle c\rangle \times \langle a^{-1}ca\rangle \cong C_3 \times C_3,$$
$$a^{-1} \cdot c \cdot a = a^{-1}ca,\ a^{-1} \cdot a^{-1}ca \cdot a = a^{-2}ca^2 = c^{-1},$$
$$\mathscr{G}_{14} = (\langle c\rangle \times \langle a^{-1}ca\rangle) \rightthreetimes \langle a\rangle \cong (C_3 \times C_3) \rightthreetimes C_4.$$

Denote the elements c, $a^{-1}ca$, and a by a, b, and c, respectively. Then

$$\mathscr{G}_{14} = \langle a, b, c \mid c^4 = a^3 = b^3 = 1,\ ab = ba,\ c^{-1}ac = b,\ c^{-1}bc = a^{-1}\rangle$$
$$= (\langle a\rangle \times \langle b\rangle) \rightthreetimes \langle c\rangle \cong (C_3 \times C_3) \rightthreetimes C_4.$$

By Lemmas 2.8–2.12 and Eqs (3.1)–(3.4), the groups $\mathscr{G}_1$–$\mathscr{G}_9$ are determined by their endomorphism semigroups in the class of all groups.

4. ON ENDOMORPHISMS OF $\mathscr{G}_{10}$ AND $\mathscr{G}_{11}$

In this section, we shall prove the following theorem.

Theorem 4.1. *The groups*

$$\mathscr{G}_{10} = \langle a, c, d \mid a^4 = c^3 = d^3 = 1,\ a^{-1}cac = 1,\ a^{-1}dad = 1,$$
$$d^{-1}c^{-1}dc = 1,\ (a^{-1}c)^2a^{-2} = 1,\ c^{-1}da^{-1}dc^{-1}a = 1\rangle$$

and

$$\mathscr{G}_{11} = \langle a, b, c, d \mid a^2 = b^2 = c^3 = d^3 = 1,\ (ac)^2 = 1,\ (ad)^2 = 1,$$
$$(ba)^2 = 1,\ c^{-1}bcb = 1,\ d^{-1}bdb = 1,\ d^{-1}c^{-1}dc = 1\rangle$$

are determined by their endomorphism semigroups in the class of all groups.

Proof. Let G be a group and G_1, G_2, K be subgroups of G such that G decomposes as follows:

$$G = (G_1 \times G_2) \leftthreetimes K = G_1 \leftthreetimes (G_2 \leftthreetimes K) = G_2 \leftthreetimes (G_1 \leftthreetimes K). \tag{4.1}$$

Denote by x, x_1, and x_2 the projections of G onto its subgroups K, $G_1 \leftthreetimes K$, and $G_2 \leftthreetimes K$, respectively. Then

$$\mathrm{Im}\,x = K,\ \mathrm{Im}\,x_1 = G_1 \leftthreetimes K,\ \mathrm{Im}\,x_2 = G_2 \leftthreetimes K,$$

$$\mathrm{Ker}\,x = G_1 \times G_2,\ \mathrm{Ker}\,x_1 = G_2,\ \mathrm{Ker}\,x_2 = G_1.$$

Assume that G^* is another group such that the endomorphism semigroups of G and G^* are isomorphic and x^*, x_1^*, and x_2^* correspond to x, x_1, and x_2 in this isomorphism. Under these assumptions the group G^* decomposes similarly to (4.1) [7, Theorems 2.1 and 3.1], i.e.,

$$G^* = (G_1^* \times G_2^*) \leftthreetimes K^* = G_1^* \leftthreetimes (G_2^* \leftthreetimes K^*) = G_2^* \leftthreetimes (G_1^* \leftthreetimes K^*),$$

$$\mathrm{Im}\,x^* = K^*,\ \mathrm{Im}\,x_1^* = G_1^* \leftthreetimes K^*,\ \mathrm{Im}\,x_2^* = G_2^* \leftthreetimes K^*,$$
$$\mathrm{Ker}\,x^* = G_1^* \times G_2^*,\ \mathrm{Ker}\,x_1^* = G_2^*,\ \mathrm{Ker}\,x_2^* = G_1^*.$$

Let us apply this result to the following group $G = G(n)$:

$$G(n) = \langle a, c, d \mid a^{2^n} = c^3 = d^3 = 1,\ cd = dc,\ a^{-1}ca = c^{-1},$$
$$a^{-1}da = d^{-1}\rangle = (\langle c\rangle \times \langle d\rangle) \leftthreetimes \langle a\rangle = \langle c\rangle \leftthreetimes (\langle d\rangle \leftthreetimes \langle a\rangle)$$
$$= \langle d\rangle \leftthreetimes (\langle c\rangle \leftthreetimes \langle a\rangle) \cong (C_3 \times C_3) \leftthreetimes C_{2^n}.$$

Hence

$$\mathrm{Im}\,x = K = \langle a\rangle \cong C_{2^n},$$
$$\mathrm{Im}\,x_1 = \langle a, c \mid a^{2^n} = c^3 = 1,\ a^{-1}ca = c^{-1}\rangle = \langle c\rangle \leftthreetimes \langle a\rangle,$$
$$\mathrm{Im}\,x_2 = \langle a, d \mid a^{2^n} = d^3 = 1,\ a^{-1}da = d^{-1}\rangle = \langle c\rangle \leftthreetimes \langle a\rangle.$$

By Lemma 2.3,

$$\mathrm{End}(\mathrm{Im}\,x_i) \cong K(x_i) \cong K(x_i^*) \cong \mathrm{End}(\mathrm{Im}\,x_i^*). \tag{4.2}$$

From Lemma 2.12 (in the case of $p = 3,\ n = 1,\ v = 2^n$) and from (4.2) we have that

$$K^* = \mathrm{Im}\,x^* = \langle a_0\rangle,$$

$$\mathrm{Im}\,x_1^* = \langle a_0, c_0 \mid a_0^{2^n} = c_0^3 = 1,\ a_0^{-1}c_0a_0 = c_0^{-1}\rangle = \langle c_0\rangle \rtimes \langle a_0\rangle,$$
$$\mathrm{Im}\,x_2^* = \langle a_0, d_0 \mid a_0^{2^n} = d_0^3 = 1,\ a_0^{-1}d_0a_0 = d_0^{-1}\rangle = \langle d_0\rangle \rtimes \langle a_0\rangle,$$
$$G_1^* = \langle c_0\rangle,\ G_2^* = \langle d_0\rangle.$$

Therefore,

$$G^* = \langle a_0, c_0, d_0 \mid a_0^{2^n} = c_0^3 = d_0^3 = 1,\ c_0d_0 = d_0c_0,$$
$$a_0^{-1}c_0a_0 = c_0^{-1},\ a_0^{-1}d_0a_0 = d_0^{-1}\rangle.$$

Hence $G^* \cong G(n)$ and the group $G(n)$ is determined by its endomorphism semigroup in the class of all groups.

By the description of the groups $\mathscr{G}_{10}$ and $\mathscr{G}_{11}$, we have

$$\mathscr{G}_{10} \cong G(2) \text{ and } \mathscr{G}_{11} \cong C_2 \times G(1).$$

We conclude from Lemmas 2.8 and 2.9 that the groups $\mathscr{G}_{10}$ and $\mathscr{G}_{11}$ are determined by their endomorphism semigroups in the class of all groups. The theorem is proved.

Hence part (1) of Theorem 1.1 is proved.

5. ON ENDOMORPHISMS OF $\mathscr{G}_{12}$

Let us consider the group

$$\mathscr{G}_{12} = \langle a, b, c \mid c^9 = b^2 = a^2 = 1,\ ab = ba,\ c^{-1}ac = b,\ c^{-1}bc = ab\rangle$$
$$= (\langle a\rangle \times \langle b\rangle) \rtimes \langle c\rangle \cong (C_2 \times C_2) \rtimes C_9.$$

In this section, we shall prove the following theorem:

Theorem 5.1. *The endomorphism semigroup of a group G is isomorphic to the endomorphism semigroup of the group $\mathscr{G}_{12}$ if and only if $G = \mathscr{G}_{12}$ or*

$$G = \langle a, b, c \mid c^9 = a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1},\ c^{-1}bc = a, c^{-1}ac = ab\rangle.$$

Proof. The group $\mathscr{G}_{12}$ is a Schmidt group, i.e., a non-nilpotent finite group in which each proper subgroup is nilpotent. The structure of the Schmidt groups is well known (see [15,16]). In [8] and [11], the endomorphisms of Schmidt groups are characterized. To prove the theorem, we present a summary of necessary results on Schmidt groups.

Each Schmidt group G can be described by three parameters p, q, and v, where p and q are different primes and v is a natural number, $v \geq 1$. A Schmidt group is not uniquely determined by its parameters. Fix parameters p, q, and v and denote by $\mathscr{S}$ the class of all Schmidt groups that have these parameters. There exist a group $G_{\max}$ of the maximal order and a group $G_{\min}$ of the minimal order in the class $\mathscr{S}$. The groups $G_{\max}$ and $G_{\min}$ are uniquely determined up to the isomorphism. A group G belongs to $\mathscr{S}$ if and only if it is isomorphic to the factor-group $G_{\max}/M$, where M is a subgroup of the second derived group $G''_{\max}$ of $G_{\max}$. Clearly, $G_{\min} \cong G_{\max}/G''_{\max}$. Let us give the description of $G_{\min}$.

Assume that $\psi(x)$ is an arbitrary irreducible normalized divisor of the polynomial

$$\frac{x^q - 1}{x - 1} = x^{q-1} + x^{q-2} + \ldots + x + 1 \in \mathbb{Z}_p[x].$$

Suppose that the degree of ψ is u. Denote by $\overline{\mathbb{Z}}_p[x]$ the residue class ring $\mathbb{Z}_p[x]/\psi(x)\mathbb{Z}_p[x]$. Then

$$G_{\min} = \{(i; f(x)) \mid i \in \mathbb{Z}_{q^v},\ f(x) \in \overline{\mathbb{Z}}_p[x]\},\ |G_{\min}| = q^v p^u,$$

and the composition rule in $G_{\min}$ is

$$(i; f(x)) \cdot (j; g(x)) = (i + j; f(x) + x^i \cdot g(x))$$

([15], Proposition 7). If u is odd, then $G_{\max} = G_{\min}$ and all Schmidt groups with parameters p, q, and v are isomorphic. If u is even, say $u = 2t$, then $|G_{\max}| = q^v p^{u+t}$ and $G_{\max}$ can be given by generators and generating relations in which the coefficients of the polynomial $\psi(x)$ are used ([16], Proposition 3).

Fix now the parameters p, q, and v as follows:

$$p = 2,\ q = 3,\ v = 2.$$

Then

$$\frac{x^q - 1}{x - 1} = \frac{x^3 - 1}{x - 1} = x^2 + x + 1 \in \mathbb{Z}_2[x].$$

Since the polynomial $x^2 + x + 1$ is irreducible in $\mathbb{Z}_2[x]$, we have

$$\psi(x) = x^2 + x + 1,\ u = 2t = 2,\ t = 1,$$

$$\overline{\mathbb{Z}}_2[x] = \mathbb{Z}_2[x]/\psi(x)\mathbb{Z}_2[x] = \{ax + b \mid a, b \in \mathbb{Z}_2\},$$

$$|G_{\min}| = q^v p^u = 3^2 2^2 = 36,\ |G_{\max}| = q^v p^{u+t} = 3^2 2^3 = 72.$$

It follows that

$$\mathscr{S} = \{G_{\min}, G_{\max}\}. \tag{5.1}$$

The group $G_{\min}$ consists of pairs $(i; f(x))$, where $i \in \mathbb{Z}_9$ and $f(x) \in \overline{\mathbb{Z}}_2[x]$. Denote $c = (1; 0)$, $a = (0; x)$, and $b = (0; 1)$. Then $G_{\min}$ can be given as follows:

$$G_{\min} = \langle a, b, c \mid c^9 = b^2 = a^2 = 1,\ ab = ba,\ c^{-1}ac = b, c^{-1}bc = ab\rangle$$

$$= (\langle a \rangle \times \langle b \rangle) \rtimes \langle c \rangle \cong (C_2 \times C_2) \rtimes C_9.$$

Therefore,

$$G_{\min} = \mathscr{G}_{12}. \tag{5.2}$$

The group $G_{\max}$ is given as follows ([16, Proposition 3]):

$$G_{\max} = \langle a, b, c \mid c^9 = a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1},$$

$$c^{-1}bc = a, c^{-1}ac = ab\rangle. \tag{5.3}$$

The endomorphism semigroups of $G_{\max}$ and $G_{\min}$ are isomorphic ([8], Theorem 4.4):

$$\mathrm{End}(G_{\max}) \cong \mathrm{End}(G_{\min}). \tag{5.4}$$

Let G^* be a group such that $\mathrm{End}(G^*) \cong \mathrm{End}(\mathscr{G}_{12}) \cong \mathrm{End}(G_{\min})$. By [11], Theorem 3.2, the group G^* is also a Schmidt group with the same parameters as $G_{\min}$, i. e., $G^* \in \mathscr{S}$. Therefore, $G^* \cong G_{\min} \cong \mathscr{G}(12)$ or $G^* \cong G_{\max}$. Isomorphism (5.4) implies the statement of the theorem. The theorem is proved and so is part (2) of Theorem 1.1.

6. ON ENDOMORPHISMS OF $\mathscr{G}_{13}$

Let us consider the group

$$\mathscr{G}_{13} = \langle d\rangle \times ((\langle a\rangle \times \langle b\rangle) \leftthreetimes \langle c\rangle) \cong C_3 \times A_4.$$

In this section, we shall find all groups G such that $\mathrm{End}(G) \cong \mathrm{End}(\mathscr{G}_{13})$.

Similarly to the previous section, there exist only two Schmidt groups with the parameters $p=2,\ q=3$, and $v=1$. They are

$$G_{\min} = \langle a, b, c \mid c^3 = b^2 = a^2 = 1,\ ab = ba,\ c^{-1}ac = b,\ c^{-1}bc = ab\rangle = A_4,$$

$$G_{\max} = \langle a, b, c \mid c^3 = a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1},\ c^{-1}ac = b,\ c^{-1}bc = ab\rangle.$$

Taking in the presentation of the binary tetrahedral group $B = \langle \alpha, \beta \mid \beta^3 = 1,\ \alpha\beta\alpha = \beta\alpha\beta\rangle$ new generators $c=\beta,\ b=\alpha\beta^{-1},\ a=\alpha\beta\alpha$, we see that B coincides with $G_{\max}$. Since the second commutator of B is $B^{''} = \langle a^2\rangle$ and $G_{\min} \cong G_{\max}/G^{''}_{\max}$, we have

$$A_4 = G_{\min} \cong G_{\max}/G^{''}_{\max} = B/B^{''} = B/\langle a^2\rangle.$$

Let us identify $A_4 = B/\langle a^2\rangle$ and denote the elements of the factor-group $B/\langle a^2\rangle$ by $\bar{g} = g\cdot\langle a^2\rangle,\ g\in B$. Then

$$\begin{aligned} B &= \langle a, b, c \mid c^3 = a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1},\ c^{-1}ac = b,\ c^{-1}bc = ab\rangle \\ &= \langle a, b\rangle \leftthreetimes \langle c\rangle. \end{aligned} \tag{6.1}$$

$$\begin{aligned} A_4 &= \langle \bar{a}, \bar{b}, \bar{c} \mid \bar{c}^3 = \bar{b}^2 = \bar{a}^2 = 1,\ \bar{a}\bar{b} = \bar{b}\bar{a}, \bar{c}^{-1}\bar{a}\bar{c} = \bar{b},\ \bar{c}^{-1}\bar{b}\bar{c} = \bar{a}\bar{b}\rangle \\ &= (\langle \bar{a}\rangle \times \langle \bar{b}\rangle) \leftthreetimes \langle \bar{c}\rangle = (C_2\times C_2)\leftthreetimes C_3. \end{aligned} \tag{6.2}$$

Lemma 6.1. *Let G^* be a group. Then $\mathrm{End}(G^*) \cong \mathrm{End}(A_4)$ if and only if $G^* = A_4$ or $G^* = B$. The map $T : \mathrm{End}(B) \longrightarrow \mathrm{End}(B/B^{''}) = \mathrm{End}(A_4)$ defined by*

$$\tau T = \overline{\tau},\ \tau\in\mathrm{End}(B),\ \bar{g}\overline{\tau} = \overline{g\tau},\ g\in B,$$

is the isomorphism of semigroups.

Proof. The proof of the first statement is the same as in the last part of the proof of Theorem 5.1. The second statement of the lemma follows from isomorphism (5.4), which is described in ([8], Theorem 4.4). The lemma is proved.

Lemma 6.2. $\mathrm{End}(C_3\times B) \cong \mathrm{End}(C_3\times A_4)$.

Proof. Assume that $C_3 = \langle d\rangle$. By (6.1) and (6.2),

$$C_3\times B = \langle d\rangle \times (\langle a, b\rangle \leftthreetimes \langle c\rangle), \tag{6.3}$$

$$C_3\times A_4 = (C_3\times B)/\langle a^2\rangle = \langle d\rangle \times (\langle \bar{a}, \bar{b}\rangle \leftthreetimes \langle \bar{c}\rangle), \tag{6.4}$$

where $\bar{d} = d\cdot\langle a^2\rangle$ is identified with d.

The groups B and A_4 satisfy the following two properties:

(1) each 3-element of A_4 has the form $\bar{g}$, where g is a 3-element of B and the numbers of 3-elements of groups A_4 and B coincide;

(2) if g is a 3-element of B and $g\neq 1$, then $\langle g\rangle \cong \langle \bar{g}\rangle \cong C_3$ and $C_B(g) = \langle g\rangle,\ C_{A_4}(\bar{g}) = \langle \bar{g}\rangle$.

In view of (6.3) and (6.4), each endomorphism T of $C_3 \times B$ has the form

$$T : \begin{cases} d \longmapsto d^i \cdot g \\ a \longmapsto a\tau \\ b \longmapsto b\tau \\ c \longmapsto c\tau \cdot d^k \end{cases},$$

where $i, k \in \mathbb{Z}_3,\ \tau \in \mathrm{End}(B)$, and g is a 3-element of B such that it commutes with $c\tau$, i.e. (a) if $c\tau = 1$, then g is an arbitrary 3-element of B; (b) if $c\tau \neq 1$, then, by property (2), $g = (c\tau)^j$, where $j \in \mathbb{Z}_3$. Similarly, each endomorphism $\overline{T}$ of $C_3 \times A_4$ has the form

$$\overline{T} : \begin{cases} d \longmapsto d^i \cdot \bar{g} \\ \bar{a} \longmapsto \bar{a}\bar{\tau} = \overline{a\tau} \\ \bar{b} \longmapsto \bar{b}\bar{\tau} = \overline{b\tau} \\ c \longmapsto \bar{c}\bar{\tau} \cdot d^k = \overline{c\tau} \cdot d^k \end{cases},$$

where $i, k \in \mathbb{Z}_3,\ \tau \in \mathrm{End}(B)$, and g is a 3-element of B such that it commutes with $c\tau$. It follows that

$$|\mathrm{End}(C_3 \times B)| \cong |\mathrm{End}(C_3 \times A_4)|,$$

$$\bar{g}\bar{T} = \overline{gT} \text{ for each } g \in C_3 \times B$$

and it is easy to see that the map $T \longmapsto \overline{T}$ gives an isomorphism $\mathrm{End}(C_3 \times B) \cong \mathrm{End}(C_3 \times A_4)$. The lemma is proved.

Theorem 6.1. *The endomorphism semigroup of a group G is isomorphic to the endomorphism semigroup of the group $\mathscr{G}_{13} = C_3 \times A_4$ if and only if $G = C_3 \times A_4$ or $G = C_3 \times B$.*

Proof. Let G be a group such that

$$\mathrm{End}(G) \cong \mathrm{End}(\mathscr{G}_{13}) \cong \mathrm{End}(C_3 \times A_4).$$

Since $\mathscr{G}_{13}$ is finite, G is finite, too ([1], Theorem 2). By Lemma 2.12, the group G splits into a direct product $G = C \times D$ such that

$$\mathrm{End}(C) \cong \mathrm{End}(C_3),\ \mathrm{End}(D) \cong \mathrm{End}(A_4).$$

By Lemmas 2.8 and 6.1, $C \cong C_3$ and $D \cong A_4$ or $D \cong B$. It follows that $G \cong C_3 \times A_4$ or $G \cong C_3 \times B$. By Lemma 6.2, the statement of the theorem is true. The theorem is proved and so is part 3 of Theorem 1.1.

7. ON ENDOMORPHISMS OF $\mathscr{G}_{14}$

In this section, we shall consider the group

$$\mathscr{G}_{14} = \langle a, b, c \mid c^4 = a^3 = b^3 = 1,\ ab = ba,\ c^{-1}ac = b,\ c^{-1}bc = a^{-1} \rangle$$

$$= (\langle a \rangle \times \langle b \rangle) \leftthreetimes \langle c \rangle \cong (C_3 \times C_3) \leftthreetimes C_4. \tag{7.1}$$

Our aim is to find all groups G such that $\mathrm{End}(G) \cong \mathrm{End}(\mathscr{G}_{14})$.

Lemma 7.1. *The endomorphisms of* $\mathscr{G}_{14}$ *are the zero-endomorphism and the following maps*:

$$y: \begin{cases} c \longmapsto cb^t a^u \\ b \longmapsto b^l a^{-k} \\ a \longmapsto b^k a^l \end{cases} \qquad t, u, k, l \in \mathbb{Z}_3; \tag{7.2}$$

$$y: \begin{cases} c \longmapsto c^3 b^t a^u \\ b \longmapsto b^{-l} a^{k} \\ a \longmapsto b^k a^l \end{cases} \qquad t, u, k, l \in \mathbb{Z}_3; \tag{7.3}$$

$$y: \begin{cases} c \longmapsto c^2 b^t a^u \\ b \longmapsto 1 \\ a \longmapsto 1 \end{cases} \qquad t, u \in \mathbb{Z}_3.$$

The maps (7.2) *and* (7.3), *where* $(k, l) \neq (0, 0)$, *are automorphisms and*

$$|\mathrm{End}(\mathscr{G}_{14})| = 172, \ |\mathrm{Aut}(\mathscr{G}_{14})| = 144.$$

Each endomorphism of $\mathscr{G}_{14}$ is induced by the images of the generators, and therefore, to prove the lemma, it is necessary to find conditions under which these images preserve the defining relations of $\mathscr{G}_{14}$. These easy calculations are omitted.

Consider the group

$$G_0 = \langle a, b, c, d \mid c^4 = a^3 = b^3 = d^3 = 1, \ ab = bad, \ c^{-1}ac = b,$$
$$c^{-1}bc = a^{-1}, \ cd = dc, \ ad = da, \ bd = db \rangle.$$

Clearly, $Z(G_0) = \langle d \rangle$ and

$$G_0/\langle d \rangle = G_0/Z(G_0) \cong \mathscr{G}_{14}. \tag{7.4}$$

Lemma 7.2. $\mathrm{End}(G_0) \cong \mathrm{End}(\mathscr{G}_{14})$.

Proof. Immediate calculations show that the endomorphisms of G_0 are the zero-endomorphism and the following maps:

$$y: \begin{cases} c \longmapsto cb^t a^u d^{(u+t)^2} \\ b \longmapsto b^l a^{-k} d^w \\ a \longmapsto b^k a^l d^r \\ d \longmapsto d^{k^2+l^2} \end{cases} \qquad t, u, k, l \in \mathbb{Z}_3; \begin{cases} w = k(u+l+t) - lt \\ r = k(u-l-t) - lt \end{cases}; \tag{7.5}$$

$$y: \begin{cases} c \longmapsto c^3 b^t a^u d^{-(u-t)^2} \\ b \longmapsto b^{-l} a^k d^w \\ a \longmapsto b^k a^l d^r \\ d \longmapsto d^{-(k^2+l^2)} \end{cases} \qquad t, u, k, l \in \mathbb{Z}_3; \begin{cases} w = k(u-t) - l(t-k+u) \\ r = k(u+t) - l(t+k-u) \end{cases}; \tag{7.6}$$

$$y: \begin{cases} c \longmapsto c^2 b^t a^u d^{-ut} \\ b \longmapsto 1 \\ a \longmapsto 1 \\ d \longmapsto 1 \end{cases} \qquad t, u \in \mathbb{Z}_3. \tag{7.7}$$

If $\tau \in \mathrm{End}(G_0)$, then the map $\bar{\tau} : G_0/Z(G_0) \longrightarrow G_0/Z(G_0)$, defined by

$$(g \cdot Z(G_0)\bar{\tau} = (g\tau) \cdot Z(G_0),\ g \in G_0,$$

is an endomorphism of $G_0/Z(G_0)$. In view of Lemma 7.1, isomorphism (7.4) and endomorphisms (7.5)–(7.7), the map

$$T : \mathrm{End}(G_0) \longrightarrow \mathrm{End}(G_0/Z(G_0)),\ \tau T = \bar{\tau},\ \tau \in \mathrm{End}(G_0)$$

is bijective. Since $(\tau_1\tau_2)T = (\tau_1 T)(\tau_2 T)$, the map T is an isomorphism. Therefore, $\mathrm{End}(G_0) \cong \mathrm{End}(G_0/Z(G_0)) \cong \mathrm{End}(\mathscr{G}_{14})$. The lemma is proved.

By (7.1), we have $\mathscr{G}_{14} = (\langle a\rangle \times \langle b\rangle) \rtimes \langle c\rangle$. Denote by x the projection of $\mathscr{G}_{14}$ onto its subgroup $\langle c\rangle$. Then $x \in I(\mathscr{G}_{14})$ and

$$\mathrm{Ker}\, x = \langle a\rangle \times \langle b\rangle \cong C_3 \times C_3,\ \mathrm{Im}\, x = \langle c\rangle \cong C_4. \tag{7.8}$$

Lemma 7.3. *The projection x given by* (7.8) *satisfies the following properties*:
1^0 $K(x) \cong \mathrm{End}(C_4)$;
2^0 $H(x) = \{0\}$;
3^0 $J(x) = \{0\}$;
4^0 $D(x) \cong C_8$;
5^0 $|[x]| = 9$;
6^0 $y \in [x],\ y \neq x \implies K(x) \cap K(y) = \{0\}$;
7^0 $V(x)$ *has a subgroup isomorphic to* $\mathscr{G}_{14}$;
8^0 $V(x) = \mathscr{A} \rtimes D(x)$, *where* $\mathscr{A} \cong C_3 \times C_3$.

Proof. By Lemma 2.3, $K(x) \cong \mathrm{End}(\mathrm{Im}\, x) \cong \mathrm{End}(C_4)$ and property 1^0 is true. Lemma 2.4 and (7.8) imply property 2^0. Lemma 2.5 and (7.8) imply that an endomorphism z of $\mathscr{G}_{14}$ belongs to $J(x)$ if and only if $cz = 1$ and $\langle a, b\rangle z \subset \langle a, b\rangle$, i.e., by Lemma 7.1, $z = 0$. Hence property 3^0 holds.

By Lemma 2.6, $D(x)$ consists of automorphisms y of $\mathscr{G}_{14}$ such that

$$\langle a, b\rangle y = \langle a, b\rangle,\ cy = c.$$

In view of Lemma 7.1,

$$D(x) = \{y \mid cy = c,\ by = b^l a^{-k},\ ay = b^k a^l;\ k, l \in \mathbb{Z}_3,\ (k, l) \neq (0, 0)\}.$$

Therefore, $|D(x)| = 8$. Choose $z \in D(x)$ as follows:

$$cz = c,\ bz = ba^{-1},\ az = ba. \tag{7.9}$$

Then

$$z^2 : \begin{cases} c \longmapsto c \\ b \longmapsto a \\ a \longmapsto b^{-1} \end{cases} \qquad z^4 : \begin{cases} c \longmapsto c \\ b \longmapsto b^{-1} \\ a \longmapsto a^{-1} \end{cases},$$

i.e., z is an element of order 8, and, therefore, $D(x) = \langle z\rangle \cong C_8$ and property 4^0 holds.

Using Lemma 7.1, it is easy to check that $I(\mathscr{G}_{14})$ consists of 0, 1 and the maps

$$y_{t,u} : \begin{cases} c \longmapsto cb^t a^u \\ b \longmapsto 1 \\ a \longmapsto 1 \end{cases} \qquad t, u \in \mathbb{Z}_3.$$

Since $x = y_{0,0}$ and $y_{t,u}x = x,\ xy_{t,u} = y_{t,u}$, we have

$$[x] = \{y_{t,u} \mid t, u \in \mathbb{Z}_3\},\ |[x]| = 9.$$

Choose $y_{t,u} \in [x]$, $y_{t,u} \neq x$, i. e., $(t, u) \neq (0,0)$. Since $\mathrm{Im}\,x = \langle c\rangle$ and $\mathrm{Im}\,y_{t,u} = \langle cb^t a^u\rangle$ are Sylow 2-subgroups of $\mathscr{G}_{14}$, there exists $g \in \mathscr{G}_{14}$ such that $\mathrm{Im}\,y_{t,u} = \langle g^{-1}cg\rangle$. We have $\mathrm{Im}\,x \cap \mathrm{Im}\,y_{t,u} = \langle 1\rangle$, because $C_{\mathscr{G}_{14}}(c^2) = \langle c\rangle$. It follows from here and Lemma 2.3 that $K(x) \cap K(y_{t,u}) = \{0\}$. Properties 5^0 and 6^0 are proved.

By Lemma 2.7, $\widehat{\mathscr{G}_{14}} = \{\widehat{g} \mid g \in \mathscr{G}_{14}\} \subset V(x)$. Since $Z(\mathscr{G}_{14}) = \{1\}$, we have $\mathscr{G}_{14} \cong \widehat{\mathscr{G}_{14}} \subset V(x)$ and property 7^0 is true.

Since an automorphism y of $\mathscr{G}_{14}$ belongs to $V(x)$ if and only if $g^{-1}\cdot gy \in \mathrm{Ker}\,x$ for each $g \in G$, Lemma 7.1 implies that

$$V(x) = \{y : \begin{cases} c \longmapsto cb^s a^t \\ b \longmapsto b^l a^{-k} \\ a \longmapsto b^k a^l \end{cases} \mid k, l \in \mathbb{Z}_3,\ (k,l) \neq (0,0)\},$$

i. e.,

$$|V(x)| = 8 \cdot 9 = 72$$

and the Sylow 3-subgroup of $V(x)$ is

$$\mathscr{A} = \{z_{i,j} \mid i, j \in \mathbb{Z}_3\} = \{\widehat{b^i a^j} \mid i, j \in \mathbb{Z}_3\} \cong C_3 \times C_3,$$

where

$$z_{i,j} : \begin{cases} c \longmapsto cb^i a^j \\ b \longmapsto b \\ a \longmapsto a \end{cases} .$$

The Sylow 2-subgroup of $V(x)$ is $D(x) = \langle z\rangle \cong C_8$, where z is given by (7.9). We have

$$c(z^{-1}z_{i,j}z) = (cb^i a^j)z = c(ba^{-1})^i (ba)^j = cb^{i+j}a^{j-i},$$

$$b(z^{-1}z_{i,j}z) = (b^{-1}a^{-1})(z_{i,j}z) = (b^{-1}a^{-1})z = b^{-1}ab^{-1}a^{-1} = b^{-2} = b,$$

$$a(z^{-1}z_{i,j}z) = (ba^{-1})(z_{i,j}z) = (ba^{-1})z = ba^{-1}b^{-1}a^{-1} = a,$$

$$z^{-1}z_{i,j}z = z_{i+j,\,j-i},$$

i.e., $\mathscr{A} \lhd V(x)$ and $V(x) = \mathscr{A} \leftthreetimes D(x)$. Property 8^0 is true. The lemma is proved.

Lemma 7.4. *If G is a finite group and there exists $x \in I(G)$ such that it satisfies properties* 1^0–8^0 *of Lemma* 7.3, *then G is isomorphic to $\mathscr{G}_{14}$ or G_0.*

Proof. Let G be a finite group such that there exists $x \in I(G)$ that satisfies properties 1^0–8^0 of Lemma 7.3.

By Lemma 2.1, we have $G = \mathrm{Ker}\,x \leftthreetimes \mathrm{Im}\,x$. Property 1^0 and Lemma 2.3 imply that $\mathrm{End}(\mathrm{Im}\,x) \cong \mathrm{End}(C_4)$. In view of Lemma 2.8, $\mathrm{Im}\,x \cong C_4$ and there exists $c \in G$ such that

$$G = M \leftthreetimes \langle c\rangle,\ \mathrm{Im}\,x = \langle c\rangle \cong C_4,\ M = \mathrm{Ker}\,x.$$

By Lemma 2.4, each $y \in H(x)$ is induced by a homomorphism $y : \mathrm{Im}\,x \longrightarrow \mathrm{Ker}\,x = M$. Property 2^0 implies that each such homomorphism is zero, i.e., M is a $2'$-group. Hence $\mathrm{Im}\,x = \langle c\rangle$ is a Sylow 2-subgroup of G. In view of Lemmas 2.1 and 2.2, $|[x]|$ is equal to the number of Sylow 2-subgroups of G. Since all Sylow 2-subgroups of G are conjugate, we have $|[x]| = [G : C_G(c)] = [M : C_M(c)]$. By property 5^0, $[M : C_M(c)] = 9$. If $g \in C_M(c)$, then $\widehat{g} \in D(x)$. Since $D(x)$ is a 2-group (property 4^0) and M is a $2'$-group, we have $\widehat{g} = 1$, i. e., $g \in Z(G)$ and

$$C_M(c) \subset Z(G),\ C_M(c) \lhd G,\ [M : C_M(c)] = |M/C_M(c)| = 9. \tag{7.10}$$

It follows that all $\{2,3\}'$-elements of G belong into the centre of G, and, therefore, G splits into a direct product

$$G = G_1 \times G_2,$$

where G_1 and G_2 are a $\{2, 3\}'$-subgroup and a $\{2, 3\}$-subgroup of G, respectively. Denote by z the projection of G onto its subgroup G_1. Then $z \in J(x)$ and property 3^0 implies $z = 0$. Therefore, $G_1 = \langle 1 \rangle$, G is a $\{2, 3\}$-group and M is a 3-group.

Since $\mathrm{Im}\, x$ is Abelian, $\widehat{g} \in V(x)$ for each $g \in G$. Therefore, by (7.10) and properties 4^0, 8^0, we get

$$M/C_M(c) \cong C_3 \times C_3. \tag{7.11}$$

Our next aim is to prove that

$$g^3 = 1 \text{ for each } g \in M. \tag{7.12}$$

Assume that k is the smallest positive integer such that $g^{3^k} = 1$ for each $g \in M$. To prove (7.12), it is necessary to show that $k = 1$. To obtain a contradiction, suppose that $k > 1$. Define the map $y : G = M \leftthreetimes \langle c \rangle \longrightarrow G$ as follows:

$$(c^i g)y = c^i \cdot g^{1+3^{k-1}},\ g \in M,\ i \in \mathbb{Z}_4.$$

To prove that y is an endomorphism of G choose $c^i g, c^j h \in G$ $(i, j \in \mathbb{Z}_4;\ g, h \in M)$. Using Lemma 2.13 and (7.11), we get

$$\begin{aligned}(c^i g \cdot c^j h)y &= (c^{i+j} \cdot (c^{-j} g c^j) \cdot h)y = c^{i+j} \cdot ((c^{-j} g c^j) \cdot h)^{1+3^{k-1}} \\ &= c^{i+j} \cdot (c^{-j} g^{1+3^{k-1}} c^j) \cdot h^{1+3^{k-1}} \cdot [h, c^{-j} g c^j]^{3^{k-1} \cdot (1+3^{k-1})/2} \\ &= c^{i+j} \cdot (c^{-j} g^{1+3^{k-1}} c^j) \cdot h^{1+3^{k-1}} \cdot [h^{3^{k-1}}, c^{-j} g c^j]^{(1+3^{k-1})/2} \\ &= c^{i+j} \cdot (c^{-j} g^{1+3^{k-1}} c^j) \cdot h^{1+3^{k-1}} = c^i g^{1+3^{k-1}} \cdot c^j h^{1+3^{k-1}} \\ &= (c^i g)y \cdot (c^j h)y.\end{aligned}$$

Hence y is an endomorphism of G. Let us find y^3:

$$(c^i g)y^3 = (c^i \cdot g^{1+3^{k-1}})y^2 = (c^i \cdot g^{(1+3^{k-1})^2})y = c^i \cdot g^{(1+3^{k-1})^3} = c^i g,$$

i.e., $y^3 = 1$ and y is an automorphism of G. Clearly, $y \in D(x)$. Property 4^0 implies that $y = 1$ and, therefore, $g^{3^{k-1}} = 1$ for each $g \in M$. This contradicts the choice of k. It follows that the assumption $k > 1$ is false and hence $k = 1$, i.e., (7.12) holds.

Choose $g \in M \setminus C_M(c)$. Then $x\widehat{g} \in [x]$, $x\widehat{g} \neq x$, $\mathrm{Ker}\, x = \mathrm{Ker}(x\widehat{g})$ and $\mathrm{Im}(x\widehat{g}) = \langle c\widehat{g} \rangle = \langle g^{-1} c g \rangle$. By Lemma 2.3, the map y_2 defined by

$$(\mathrm{Ker}\, x)y_2 = \langle 1 \rangle,\ c y_2 = (g^{-1} c g)y_2 = g^{-1} c^2 g$$

belongs to $K(x\widehat{g})$. Clearly, $y_2 \neq 0$. If $c^2 \in Z(G)$, then $c y_2 = c^2$ and $y_2 \in K(x) \cap K(x\widehat{g})$. This contradicts property 6^0. Therefore, $c^2 \notin Z(G)$, $\widehat{c}^2 \neq 1$ and $\widehat{c}$ is an automorphism of order 4 of G. It follows from (7.10), (7.11), and Lemma 2.7 that

$$\widehat{G} = \widehat{M} \leftthreetimes \langle \widehat{c} \rangle \cong (C_3 \times C_3) \leftthreetimes C_4,\ \widehat{G} \subset V(x),\ |\widehat{G}| = 36.$$

We have also that $Z(G) = C_M(c)$. By properties 4^0 and 8^0, $|V(x)| = 72$. Hence

$$[V(x) : \widehat{G}] = 2. \tag{7.13}$$

In view of property 7^0, $V(x)$ has a subgroup $\mathscr{B}$ isomorphic to $\mathscr{G}_{14}$. We have

$$[V(x) : \mathscr{B}] = 2 \tag{7.14}$$

because $|V(x)| = 72$ and $|\mathscr{G}_{14}| = 36$. Since $D(x)$ is cyclic, property 8^0 implies that $V(x)$ has only one subgroup of index 2. Therefore, by (7.13) and (7.14),

$$\widehat{G} \cong \mathscr{G}_{14} \cong G/C_M(c). \tag{7.15}$$

It follows from (7.1) and (7.15) that there exist $a, b \in G$ such that

$$M/C_M(c) = \langle \overline{a} \rangle \times \langle \overline{b} \rangle \cong C_3 \times C_3,$$

where

$$\overline{a} = a \cdot C_M(c),\ \overline{b} = b \cdot C_M(c)$$

and

$$\overline{c}^{-1}\overline{ac} = \overline{b},\ \overline{c}^{-1}\overline{bc} = \overline{a}^{-1},\ \overline{c} = c \cdot C_M(c).$$

We can choose

$$b = c^{-1}ac.$$

Then

$$c^{-1}bc = a^{-1}h \text{ for some } h \in C_M(c).$$

Denote $d = h^2$. Then $d \in C_M(c) \subset Z(G)$ and, by (7.12), $d^2 = h$. We have

$$c^{-1}bc = a^{-1}h = a^{-1}d^2,$$

$$c^{-1} \cdot bd^{-1} \cdot c = a^{-1}d = (ad^{-1})^{-1},$$

$$c^{-1} \cdot ad^{-1} \cdot c = c^{-1}ac \cdot d^{-1} = bd^{-1}.$$

Denote the elements ad^{-1} and bd^{-1} by a and b, respectively. Then

$$c^{-1}ac = b,\ c^{-1}bc = a^{-1},\ [a, b] \in C_M(c) \subset Z(G).$$

Consider the subgroup

$$N = \langle c, a, b, [a, b] \rangle = \langle a, b, [a, b] \rangle \leftthreetimes \langle c \rangle$$

of G. Clearly, $N \lhd G$. The factor-group $G/N = (C_M(c) \cdot N)/N$ is an elementary Abelian 3-group, because $g^3 = 1$ for each $g \in M$. Assume that $N \neq G$. There exist $h \in C_M(c)$ and $L \lhd G$ such that $N \subset L$ and $G/L = \langle hL \rangle \cong C_3$. Consider the endomorphism y of G given as follows:

$$y = \varepsilon z : G \xrightarrow{\varepsilon} G/L = \langle hL \rangle \xrightarrow{z} G,\ (hL)z = a,$$

where ε is the natural homomorphism. By the construction of y, $y \neq 0$ and $y \in J(x)$. This contradicts property 3^0. Therefore, $G = N$. Since $g^3 = 1$ for each $g \in M$, we have

$$G = \langle c, a, b, [a, b] \rangle = \langle a, b, [a, b] \rangle \leftthreetimes \langle c \rangle,$$

where

$$c^4 = 1,\ a^3 = b^3 = [a, b]^3 = 1,\ c^{-1}ac = b,\ c^{-1}bc = a^{-1},\ [a, b] \in Z(G).$$

If $[a, b] = 1$, then

$$G = \langle a, b, c \mid c^4 = a^3 = b^3 = 1,\ ab = ba,\ c^{-1}ac = b,\ c^{-1}bc = a^{-1} \rangle = \mathscr{G}_{14}.$$

If $[a, b] \neq 1$, then denoting $d = [a, b]$, we have

$$G = \langle a, b, c, d \mid c^4 = a^3 = b^3 = d^3 = 1,\ ab = bad,\ c^{-1}ac = b,$$

$$c^{-1}bc = a^{-1},\ cd = dc,\ ad = da,\ bd = db \rangle = G_0.$$

The lemma is proved.

Theorem 7.1. *The endomorphism semigroup of a group G is isomorphic to the endomorphism semigroup of the group $\mathscr{G}_{14}$ if and only if $G = \mathscr{G}_{14}$ or $G = G_0$.*

Proof. Let G be a group such that

$$\mathrm{End}(G) \cong \mathrm{End}(\mathscr{G}_{14}). \tag{7.16}$$

Since $\mathscr{G}_{14}$ is finite, G is finite, too ([1], Theorem 2). By Lemma 7.3 and isomorphism (7.16), there exists $x \in I(G)$ that satisfy properties 1^0–8^0 of Lemma 7.3. Lemma 7.4 implies that G is isomorphic to $\mathscr{G}_{14}$ or G_0. It follows from here and Lemma 7.2 that the statement of the theorem is true. The theorem is proved and so is part (4) of Theorem 1.1.

Theorem 1.1 is proved.

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36. järku rühmade endomorfismidest

Alar Leibak ja Peeter Puusemp

Eksisteerib 14 mitteisomorfset 36. järku rühma. On näidatud, et nendest on ainult 11 määratud oma endomorfismipoolrühmaga kõigi rühmade klassis. Ülejäänud kolme puhul leidub igaühe jaoks parajasti üks temaga mitteisomorfne rühm, millel on sama endomorfismipoolrühm (isomorfismi täpsuseni).