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MATHEMATICS

On approximation by Blackman- and Rogosinski-type operators in Banach space

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Abstract. In this paper we introduce the Blackman- and Rogosinski-type approximation processes in an abstract Banach space setting. Historical roots of these processes go back to W. W. Rogosinski in 1926. The new definitions given use the concept of cosine operator functions. We proved that in the presented setting the Blackman- and Rogosinski-type operators possess the order of approximation, which coincides with results known in trigonometric approximation. Applications for the Fourier–Chebyshev approximation are given as well.

Key words: cosine operator function, Blackman- and Rogosinski-type approximation processes, modulus of continuity.

1. INTRODUCTION

Let X be an arbitrary (real or complex) Banach space and $[X]$ be the Banach algebra of all bounded linear operators U of X into itself. Let $\{P_k\}_{k=0}^{\infty} \subset [X]$ be a given sequence of mutually orthogonal projections, i.e. $P_j P_k = \delta_{jk} P_k$ (δ_{jk} being the Kronecker symbol). Moreover, let us assume that the sequence of projections is total, i.e. $P_k f = 0$ for all $k = 0, 1, 2, \ldots$ implies $f = 0$, and fundamental, i.e. the linear span of $\bigcup_{k=0}^{\infty} P_k(X)$ is dense in X. Then with each $f \in X$ one may associate its unique Fourier series expansion

$$f \sim \sum_{k=0}^{\infty} P_k f$$

with the Fourier partial sums operator

$$S_n f = \sum_{k=0}^{n} P_k f.$$

As we know from trigonometric Fourier approximation, the convergence of the Fourier partial sums is not guaranteed for all $f \in X$. The improvement of that situation will be given by some matrix transformation like

$$U_n f = \sum_{k=0}^{n} \Theta_k(n) P_k f.$$

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In this paper we introduce the Blackman- and Rogosinski-type operators in an abstract setting and find the order of approximation via a modulus of continuity (smoothness), which is defined by a general translation operator. The Blackman- and Rogosinski-type operators are interesting, because in a trigonometric case we are able to calculate precise values of their operator norms.

Definition 1. *A cosine operator function $T_h \in [X]$ $(h \geq 0)$ is defined by the following properties*:
(i) $T_0 = I$ (*identity operator*),
(ii) $T_{h_1} \cdot T_{h_2} = \frac{1}{2}(T_{h_1+h_2} + T_{|h_1-h_2|})$,
(iii) $\|T_h f\| \leq T\|f\|$, $0 < T-$ *not depending on* $h > 0$.

Remark 1. Let $\tau_h \in [X]$, $h \in \mathbb{R}$, be a translation operator, defined by the properties
(i) $\tau_0 = I$,
(ii) $\tau_{h_1} \cdot \tau_{h_2} = \tau_{h_1+h_2}$,
(iii) $\|\tau_h f\| \leq T\|f\|, 0 < T-$ not depending on $h \in \mathbb{R}$.
Then $T_h := \frac{1}{2}(\tau_h + \tau_{-h})$, $h \geq 0$, is a cosine operator function.

First let us consider some examples.

Classical trigonometric case. Let $X = C_{2\pi}$ be the space of 2π-periodic continuous functions and $\tau_h(f,x) = f(x+h)$, $h \in \mathbb{R}$, which obviously forms a semigroup translation with $T = 1$. Then by Remark 1

$$T_h(f,x) = \frac{1}{2}\left(f(x+h) + f(x-h)\right), \ h \geq 0 \tag{1.1}$$

forms the cosine operator function.

It is well known that the Fourier partial sums operator

$$S_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

is translation invariant, i.e. $T_h(S_n f,x) = S_n(T_h f,x)$. Moreover,

$$T_h\left(S_n f,x\right) = \frac{a_0}{2} + \sum_{k=1}^{n} \cos kh\left(a_k \cos kx + b_k \sin kx\right). \tag{1.2}$$

π-Symmetric trigonometric case. Let $X = C_{2\pi}^{-}$ be the space of π-symmetric continuous functions, i.e. we suppose that $f(2\pi - x) = f(x)$ and $f(4\pi + x) = f(x)$ for all $x \in \mathbb{R}$. For example, the functions $y = \cos kx$, $y = \sin(k+\frac{1}{2})x$, $k = 0,1,2,\dots$ belong to the space $C_{2\pi}^{-}$. Since the system $\left\{\cos kx,\ \sin(k+\frac{1}{2})x\right\}_{k=0}^{\infty}$ is orthogonal on $[-\pi,\pi]$ under a usual scalar product, we may consider the Fourier partial sums operator

$$S_n^{-}(f,x) = {\sum_{k=0}^{n}}{}' \left(a_k \cos kx + d_k \sin\left(k + \frac{1}{2}\right) x\right), \tag{1.3}$$

where $\sum'$ means that the coefficient a_0 is halved and

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt dt, \ d_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin\left(k+\frac{1}{2}\right) t dt.$$

If a function $f \in C_{2\pi}$, it is obvious that $\tau_h f \in C_{2\pi}$ as well. But it should be noted that the ordinary translation operator $\tau_h(f,x) = f(x+h)$, $h \in \mathbb{R}$, is not suitable for the π-symmetric functions, since, for example, $\tau_h\left(\sin\left(\frac{1}{2}\circ\right),x\right) = \sin\frac{1}{2}(x+h) \notin C_{2\pi}^{-}$ for some $h \in \mathbb{R}$. For the cosine operator function (1.1) we state following

Lemma 1. *Let $f \in C_{2\pi}^{-}$. Then for every $h \geq 0$ the cosine operator function yields $T_h f \in C_{2\pi}^{-}$, where $T_h := \frac{1}{2}(\tau_h + \tau_{-h})$ and τ_h is the ordinary translation operator.*

Proof. Consider $f \in C_{2\pi}^{-}$, then for all $h \geq 0$ the function $T_h f$ satisfies the conditions $f(2\pi - x) = f(x)$ and $f(4\pi + x) = f(x)$. □

Analogously to the periodic case we have

$$T_h\left(S_n^{-} f, x\right) = {\sum_{k=0}^{n}}' \left(a_k \cos kh \cos kx + d_k \cos\left(k+\frac{1}{2}\right) h \sin\left(k+\frac{1}{2}\right) x\right).$$

Fourier–Chebyshev series. For $f \in C_{[-1,1]}$ let us consider the Fourier–Chebyshev partial sums operator

$$S_n^C(f,x) = \hat{f}_C(0) + 2\sum_{k=1}^{n} \hat{f}_C(k) T_k(x),$$

where

$$\hat{f}_C(k) := \frac{1}{\pi}\int_{-1}^{1} f(u) T_k(u) \frac{du}{\sqrt{1-u^2}}$$

is the kth Fourier–Chebyshev coefficient and $T_k(u) = \cos(k \arccos u)$ is the kth Chebyshev polynomial. For this case a suitable cosine operator function (see [1,3]) is

$$T_h^C(f,x) := \frac{1}{2}\left\{f(x\cos h + \sqrt{1-x^2}\sin h) + f(x\cos h - \sqrt{1-x^2}\sin h)\right\}, \; 0 \leq h \leq \pi. \tag{1.4}$$

Since it can be verified that $T_h^C(T_k, x) = \cos kh \, T_k(x)$, we have

$$T_h^C(S_n^C f, x) = \hat{f}_C(0) + 2\sum_{k=1}^{n} \cos kh \, \hat{f}_C(k) T_k(x). \tag{1.5}$$

The paper is organized as follows.

Section 2 is devoted to preliminary notions like the modulus of continuity and the best approximation, but in an abstract setting using the cosine operator function. The main definitions of the Blackman- and Rogosinski-type approximation operators are introduced.

Section 3 treats the order of approximation by the Blackman- and Rogosinski-type approximation operators.

Section 4 concerns applications to the trigonometric approximation operators.

In Section 5 we consider a quite specific problem. Namely, it appears that the exact values of the operator norms of the Blackman- and Rogosinski-type trigonometric approximation operators can be calculated.

In Section 6 we apply the previous results to the Fourier–Chebyshev series.

2. MODULUS OF CONTINUITY, THE BEST APPROXIMATIONS, AND BLACKMAN- AND ROGOSINSKI-TYPE OPERATORS

In the present section we will define the Blackman- and Rogosinski-type approximation operators. The main idea for definitions below was inspired from the trigonometric approximation (see [2,4,5,7] and references cited there).

An abstract modulus of continuity, defined by the cosine operator function, will play an important role in our paper.

Definition 2. *The modulus of continuity of order k is defined via the cosine operator function by*

$$\omega_k(f,\delta) := \sup_{0\leq h\leq\delta} \|(T_h - I)^k f\|,\ k\in\mathbb{N}. \tag{2.1}$$

Proposition 1. *The modulus of continuity* $\omega_k(f,\delta)$ $(\omega(f,\delta) := \omega_1(f,\delta))$ *has the following properties*:

(i) $\omega(f,m\delta) \leq m(1+(m-1)T)\omega(f,\delta)$, $m\in\mathbb{N}$;

(ii) $\omega(f,\lambda\delta) \leq ([\lambda]+1)(1+[\lambda]T)\omega(f,\delta)$, $\lambda>0$, ($[\lambda]\leq\lambda$ *is the entire part of* $\lambda\in\mathbb{R}$);

(iii) $\omega_k(f,\delta) \leq (1+T)^{k-l}\omega_l(f,\delta)$, $k\geq l$ *and* $k,l\in\mathbb{N}$.

Remark 2. Let $\tau_h : X\to X$, $h\in\mathbb{R}$, be a translation operator and let us define another modulus of continuity by

$$\widetilde{\omega}_k(f,\delta) := \sup_{0\leq h\leq\delta}\left\|\left(\tau_{h/2}-\tau_{-h/2}\right)^k f\right\|,\ k\in\mathbb{N}.$$

Then by Remark 1 $T_h := \frac{1}{2}\left(\tau_h+\tau_{-h}\right)$, $h\geq 0$, defines the modulus of continuity ω_k by (2.1). Since $T_h - I = \frac{1}{2}\left(\tau_{h/2}-\tau_{-h/2}\right)^2$, we have

$$\omega_k(f,\delta) = \frac{1}{2^k}\widetilde{\omega}_{2k}(f,\delta). \tag{2.2}$$

Another quantity we need is the best approximation.

Definition 3. *Let us denote* $X_k := P_k(X)$ *and* $\Pi_n := X_0 + X_1 + \ldots + X_n = \{\sum_{k=0}^n \alpha_k P_k f \mid \alpha_k\in\mathbb{R}, f\in X\}$. *Then*

$$E_n(f) := \inf_{p_n\in\Pi_n}\|f-p_n\|$$

is the best approximation of $f\in X$ *by polynomials* $p_n\in\Pi_n$.

Remark 3. Usually we may assume that for each $f\in X$ there exists $p_n^*\in\Pi_n$ such that $\|f-p_n^*\| = E_n(f)$.

Let us define the Rogosinski- and Blackman-type operators.

Definition 4. *The Rogosinski-type operators* $R_{n,h,a} : X\to X$ *are defined via the Fourier partial sums and the cosine operator function is defined by*

$$R_{n,h,a}f := aT_h(S_n f) + (1-a)\,T_{3h}(S_n f)\ \ (h\geq 0, a\in\mathbb{R}).$$

Remark 4. The case $a=1$ leads to the original Rogosinski operator $R_{n,h} : C_{2\pi}\to C_{2\pi}$ which was introduced by W. W. Rogosinski [5] in trigonometric approximation and was afterwards elaborated by S. B. Stechkin in [6]; see also [2,7,8].

Definition 5. *The Blackman-type operators* $B_{n,h,a} : X\to X$ *are defined via the Fourier partial sums and the cosine operator function is defined by*

$$B_{n,h,a}f := aS_n f + \frac{1}{2}T_h(S_n f) + \left(\frac{1}{2}-a\right)T_{2h}(S_n f)\ \ (h\geq 0, a\in\mathbb{R}).$$

Remark 5. In Definition 5 the Blackman operator in the case $a=\frac{1}{2}$ is called the Hann operator, denoted here by $H_{n,h}$. If the Fourier partial sums operator is translation invariant, i.e. $T_h S_n = S_n T_h$, then it is easy to prove that $R_{n,h}^2 = H_{n,2h}$ and $B_{n,h,3/8} = H_{n,h}^2$.

3. ORDER OF APPROXIMATION BY BLACKMAN- AND ROGOSINSKI-TYPE OPERATORS

In this section we discuss the order of approximation of the Blackman- and Rogosinski-type operators by the modulus of continuity.

Theorem 1. *For every $f \in X$ and all $a \in \mathbb{R}$ for the Blackman-type operators $B_{n,h,a} : X \to X$ we have*

$$\|B_{n,h,a}f - f\| \leq \left(\|B_{n,h,a}\|_{[X]} + |a| + \frac{T}{2} + \left|\frac{1}{2} - a\right| T\right) E_n(f) + \frac{1}{4}\omega(f,h) + \frac{|1-2a|}{4}\omega(f,2h), \tag{3.1}$$

where the constant $T > 0$ is from Definition 1 (*assumption* (iii)).

Proof. Let $p_n^* \in \Pi_n$ be an element of the best approximation of $f \in X$. Then

$$\|B_{n,h,a}f - f\| \leq \|B_{n,h,a}f - B_{n,h,a}p_n^*\| + \|B_{n,h,a}p_n^* - f\| \leq \|B_{n,h,a}\|_{[X]}E_n(f) + \|B_{n,h,a}p_n^* - f\|. \tag{3.2}$$

Denote

$$\Theta_{h,a}f := af + \frac{1}{2}T_h f + \frac{1-2a}{2}T_{2h}f. \tag{3.3}$$

Since $S_n p_n^* = p_n^*$, Definition 5 implies

$$B_{n,h,a}p_n^* = \Theta_{h,a}p_n^*.$$

Therefore,

$$\|B_{n,h,a}p_n^* - f\| \leq \|\Theta_{h,a}p_n^* - \Theta_{h,a}f\| + \|\Theta_{h,a}f - f\|. \tag{3.4}$$

By definition of $\Theta_{h,a}$ in (3.3), using the property (iii) of the translation operator in Definition 1, we obtain

$$\|\Theta_{h,a}p_n^* - \Theta_{h,a}f\| \leq \left(|a| + \frac{T}{2} + \frac{|1-2a|}{2}T\right) E_n(f). \tag{3.5}$$

For the second term in the left-hand side of (3.4) we write

$$\Theta_{h,a}f - f = \frac{1}{2}(T_h - I)f + \frac{1-2a}{2}(T_{2h} - I)f.$$

Thus, by Definition 2 we get

$$\|\Theta_{h,a}f - f\| \leq \frac{1}{2}\omega(f,h) + \frac{1-2a}{2}\omega(f,2h).$$

Substituting all expressions above in (3.2) yields the assertion of the theorem. □

Remark 6. By properties of the modulus of continuity

$$\omega(f,h) \leq \omega(f,mh) \leq m(1+(m-1)T)\omega(f,h),\ m \in \mathbb{N}.$$

Thus, in Theorem 1 the quantities $\omega(f,h)$ and $\omega(f,2h)$ have the same order, and one may conclude that the simplest case is when $a = \frac{1}{2}$. This situation will be fixed as follows.

Corollary 1. *Let the Hann operator $H_{n,h} : X \to X$ be defined by the equation*

$$H_{n,h}f := \frac{1}{2}(S_n f + T_h(S_n f)).$$

Then for every $f \in X$ the following holds:

$$\|H_{n,h}f - f\| \leq \left(\frac{1+T}{2} + \|H_{n,h}\|_{[X]}\right) E_n(f) + \frac{1}{4}\omega(f,h).$$

The importance of the Blackman-type operators consists in the following statement.

Theorem 2. *For every $f \in X$ it holds that*

$$\|B_{n,h,5/8}f - f\| \leq \left(\frac{5}{8}(1+T) + \|B_{n,h,5/8}\|_{[X]}\right) E_n(f) + \frac{1}{4}\omega_2(f,h). \tag{3.6}$$

Proof. Similarly to the proof of Theorem 1 we proceed

$$\|B_{n,h,5/8}f - f\| \leq \|B_{n,h,5/8}\|_{[X]} E_n(f) + \|\Theta_h p_n^* - \Theta_h f\| + \|\Theta_h f - f\|, \tag{3.7}$$

where

$$\Theta_h f := \frac{5}{8} f + \frac{1}{2} T_h f - \frac{1}{8} T_{2h} f. \tag{3.8}$$

Since by Definition 1

$$(T_h - I)^2 = \frac{1}{2}\left(T_{2h} - 4T_h + 3I\right),$$

by (3.8) we have

$$f - \Theta_h f = \frac{1}{4}(T_h - I)^2 f. \tag{3.9}$$

Taking $a = \frac{5}{8}$ in (3.5), we get

$$\|\Theta_h p_n^* - \Theta_h f\| \leq \frac{5}{8}(1+T) E_n(f),$$

which together with (3.9) and (3.7) gives the assertion.

Analogous results are valid for the Rogosinski-type operators. □

Theorem 3. *For every $f \in X$, $a \in \mathbb{R}$ for the Rogosinski-type operators $R_{n,h,a} : X \to X$ it holds that*

$$\|R_{n,h,a}f - f\| \leq \left(\|R_{n,h,a}\|_{[X]} + |a|T + |1-a|T\right) E_n(f) + |a|\omega(f,h) + |1-a|\omega(f,3h). \tag{3.10}$$

Again, as by Remark 6 $\omega(f,h)$ and $\omega(f,3h)$ have the same order and $g(a) = |a| + |1-a|$ $(a \in \mathbb{R})$ has its minimum value on $[0,1]$, we specify

Corollary 2. 1. *For $0 \leq a < 1$ the following relation holds:*

$$\|R_{n,h,a}f - f\| \leq \left(\|R_{n,h,a}\|_{[X]} + T\right) E_n(f) + \omega(f,3h).$$

2. *Let us denote $R_{n,h} := R_{n,h,1}$. Then it holds that*

$$\|R_{n,h}f - f\| \leq \left(\|R_{n,h}\|_{[X]} + T\right) E_n(f) + \omega(f,h).$$

The specific value $a = \frac{9}{8}$ yields a better order of approximation.

Theorem 4. *Denote $\widetilde{R}_{n,h} = R_{n,h,9/8}$. Then we have*

$$\|\widetilde{R}_{n,h}f - f\| \leq \left(\|\tilde{R}_{n,h}\|_{[X]} + \frac{5}{4}T\right) E_n(f) + \frac{3}{2}\omega_2(f,h) + \frac{1}{2}\omega_3(f,h).$$

Corollary 3. *Using the property $\omega_k(f,\delta) \leq (1+T)^{k-l}\omega_l(f,\delta)$, $k \geq l$ of modulus of continuity, we get*

$$\|\widetilde{R}_{n,h}f - f\| \leq \left(\|\widetilde{R}_{n,h}\|_{[X]} + \frac{5}{4}T\right) E_n(f) + \frac{4+T}{2}\omega_2(f,h).$$

As we may see from Theorems 1–4 and their corollaries, Definitions 3 and 4 deduce approximation processes when the right-hand sides of the given estimates tend to zero as $n \to \infty$ and $h \to 0+$. This statement requires, among others, that the sequences of operator norms $\|B_{n,h,a}\|_{[X]}$, $\|R_{n,h,a}\|_{[X]}$ have to be uniformly bounded on n and h. From this argumentation it follows that $n \in \mathbb{N}$ and $h \geq 0$ should be related in some way. We will consider this problem in certain situations like in approximation by trigonometric polynomials or by the Fourier–Chebyshev series. However, in a general framework we may state as follows. The families of Blackman- and Rogosinski-type operators possess the representation

$$U_a = aV + W, \ V, W \in [X].$$

Then for every $a, a_0, a_1 \in \mathbb{R}$ we get

$$(a_1 - a_0)U_a = (a_1 - a)U_{a_0} + (a - a_0)U_{a_1}.$$

This equality shows that for the boundedness of $\|U_a\|_{[X]}$ for arbitrary $a \in \mathbb{R}$ it is enough to know the boundedness of $\|U_{a_0}\|_{[X]}$ and $\|U_{a_1}\|_{[X]}$ for two specific values $a_0, a_1 \in \mathbb{R}$.

4. APPROXIMATION BY TRIGONOMETRIC BLACKMAN- AND ROGOSINSKI-TYPE OPERATORS

It appears that in classical trigonometric approximation the family of the Blackman-type operators will be uniformly bounded if $h = \frac{\pi}{n+1}$. By Definition 5, using (1.2), it follows that for Blackman-type operators we have

$$B_{n,a}(f,x) := \frac{a_0}{2} + \sum_{k=1}^{n} \varphi_a\left(\frac{k}{n+1}\right)(a_k \cos kx + b_k \sin kx), \tag{4.1}$$

where

$$\varphi_a(t) = a + \frac{1}{2}\cos \pi t + \left(\frac{1}{2} - a\right)\cos 2\pi t, \, t \in [0,1], \, a \in \mathbb{R}. \tag{4.2}$$

In space $C_{2\pi}$ the ordinary translation operator $\tau_h(f,x) = f(x+h), \ h \in \mathbb{R}$, is well defined, moreover $T = 1$. The translation operator $\tau_h : C_{2\pi} \to C_{2\pi}$ produces the ordinary modulus of continuity given in Remark 2. Therefore, Theorem 1 yields

Theorem 5. *For every $f \in C_{2\pi}$ and all $a \in \mathbb{R}$ for the Blackman-type operators* (4.1) *it holds that*

$$\|B_{n,a}f - f\|_{C_{2\pi}} \leq \left(\|B_{n,a}\|_{[C_{2\pi}]} + |a| + \left|\frac{1}{2} - a\right| + \frac{1}{2}\right) E_n(f) + \frac{1}{8}\widetilde{\omega}_2\left(f, \frac{\pi}{n+1}\right)$$
$$+ \frac{|1-2a|}{8}\widetilde{\omega}_2\left(f, \frac{2\pi}{n+1}\right).$$

Remark 7. Since $|a| + |1/2 - a| \geq 1/2$ with equality for $0 \leq a \leq 1/2$, it would be interesting to consider the Blackman operators $B_{n,a}$ only in the case $0 \leq a \leq 1/2$ and in the case $a = 5/8$ due to Theorem 2.

Definition 4 with $h = \frac{\pi}{2(n+1)}$, using (1.2), yields the trigonometric Rogosinski-type operators

$$R_{n,a}(f,x) := \frac{a_0}{2} + \sum_{k=1}^{n} \Psi_a\left(\frac{k}{n+1}\right)(a_k \cos kx + b_k \sin kx), \tag{4.3}$$

where

$$\Psi_a(t) = a\cos\frac{\pi t}{2} + (1-a)\cos\frac{3\pi t}{2}, \ \ t \in [0,1], \ \ a \in \mathbb{R}. \tag{4.4}$$

Theorem 6. *For every $f \in C_{2\pi}$, $a \in \mathbb{R}$ for the Rogosinski-type operators $R_{n,a} : C_{2\pi} \to C_{2\pi}$ it holds that*

$$\|R_{n,a}f - f\|_{C_{2\pi}} \le \left(\|R_{n,a}\|_{[C_{2\pi}]} + |a| + |1-a|\right) E_n(f) + \frac{|a|}{2}\widetilde{\omega_2}\left(f, \frac{\pi}{2(n+1)}\right) + \frac{|1-a|}{2}\widetilde{\omega_2}\left(f, \frac{3\pi}{2(n+1)}\right).$$

Now we make some preparations for the calculation of the norms $\|B_{n,a}\|_{[C_{2\pi}]}$. It is known [8] that the trigonometric polynomial operator (or summability operator)

$$U_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n} \varphi\left(\frac{k}{n+1}\right)(a_k \cos kx + b_k \sin kx),$$

where $\varphi \in C_{[0,1]}, \varphi(0) = 1, \varphi(1) = 0$, transforms the space $C_{2\pi}$ into $C_{2\pi}$, and its norm $\|U_n\|_{[C_{2\pi}]} \equiv \|U_n\|_{C_{2\pi}\to C_{2\pi}}$ satisfies

$$\sup_n \|U_n\|_{[C_{2\pi}]} = \int_{-\infty}^{\infty} |s(u)|du, \tag{4.5}$$

where the kernel function $s \in L^1(\mathbb{R})$ is given by

$$s(u) = \int_0^1 \varphi(t)\cos(\pi t u)dt. \tag{4.6}$$

By (4.5) and (4.2) we find

$$\sup_n \|B_{n,a}\|_{[C_{2\pi}]} = 2\int_0^{\infty} |s_{B,a}(u)|du, \tag{4.7}$$

where by (4.2) and (4.6)

$$s_{B,a}(t) = \frac{((3-8a)t^2 + 8a)\operatorname{sinc} t}{2(1-t^2)(4-t^2)} \tag{4.8}$$

with

$$\operatorname{sinc} t := \frac{\sin(\pi t)}{\pi t}. \tag{4.9}$$

To calculate the norm in (4.7), we will follow the scheme

$$\int_0^{\infty}(\ldots) = \int_0^{t_a}(\ldots) + \int_{t_a}^{m}(\ldots) + \sum_{k=m}^{\infty}\int_k^{k+1}(\ldots).$$

Let us denote the polynomial in the numerator of (4.8) by

$$p(t) := (3-8a)t^2 + 8a. \tag{4.10}$$

It is clear that the signs of the polynomial p considerably influence the signs of the kernel $s_{B,a}$. Let us consider for this polynomial the following possibilities.

1. If $0 \le a \le \frac{3}{8}$, then $p(t) \ge 0$ for all $t \in \mathbb{R}$.
2. If $a > \frac{3}{8}$, then the unique positive zero of the polynomial p is $t_a = \sqrt{\frac{8a}{8a-3}} > 1$. Moreover, $p(t) \ge 0$, if $0 \le t \le t_a$, and $p(t) < 0$, if $t > t_a$. For the sake of technical simplification let $\frac{1}{2} \le a \le \frac{5}{8}$ (due to Theorem 2 and Corollary 1). In this case $\sqrt{\frac{5}{2}} \le t_a \le 2$.

In what follows we need some technical results in order to calculate integrals $\int_k^{k+1} s_{B,a}(t)dt$.

Lemma 2. *The kernel function* (4.8) *can be represented by*

$$s_{B,a}(t) = a\left(r\left(t-\frac{1}{2}\right)+r\left(t+\frac{1}{2}\right)\right)+\frac{1-2a}{2}\left(r\left(t-\frac{3}{2}\right)+r\left(t+\frac{3}{2}\right)\right), \tag{4.11}$$

where the original Rogosinski kernel is

$$s_{R,1} \equiv r(t) = \int_0^1 \cos\frac{\pi u}{2}\cos(\pi t u)du. \tag{4.12}$$

Proof. Since for all $b \in \mathbb{R}$ it holds that

$$r(t-b)+r(t+b) = \int_0^1 \left(\cos\pi\left(\frac{1}{2}-b\right)u+\cos\pi\left(\frac{1}{2}+b\right)u\right)\cos\pi t u\, du, \tag{4.13}$$

we get

$$r\left(t-\frac{1}{2}\right)+r\left(t+\frac{1}{2}\right) = \int_0^1 (1+\cos\pi u)\cos\pi t u\, du,$$

$$r\left(t-\frac{3}{2}\right)+r\left(t+\frac{3}{2}\right) = \int_0^1 (\cos\pi u+\cos 2\pi u)\cos\pi t u\, du.$$

Now, by (4.6) and (4.2), it is easy to verify that (4.11) is valid. □

Let us denote the modified integral sine as

$$\mathrm{Sci}(x) := \int_0^x \mathrm{sinc}(t)dt. \tag{4.14}$$

Lemma 3. *Denote* $(k \in \mathbb{Z})$

$$S_k := \frac{1}{2}\left(\mathrm{Sci}(k+2)-\mathrm{Sci}(k)\right), \tag{4.15}$$

then

$$I_{a,k} := \int_k^{k+1} s_{B,a}(t)dt = a(S_{k-1}+S_k)+\left(\frac{1}{2}-a\right)(S_{k-2}+S_{k+1}). \tag{4.16}$$

Proof. By (4.11) we have to calculate

$$J_{1,k} := \int_k^{k+1}\left(r\left(u-\frac{1}{2}\right)+r\left(u+\frac{1}{2}\right)\right)du, \tag{4.17}$$

$$J_{2,k} := \int_k^{k+1}\left(r\left(u-\frac{3}{2}\right)+r\left(u+\frac{3}{2}\right)\right)du. \tag{4.18}$$

Using the definition of the sinc-function and (4.13), we get

$$r(t-b)+r(t+b)=\frac{1}{2}\left(\operatorname{sinc}\left(t+b-\frac{1}{2}\right)+\operatorname{sinc}\left(t-b+\frac{1}{2}\right)+\operatorname{sinc}\left(t-b-\frac{1}{2}\right)+\operatorname{sinc}\left(t+b+\frac{1}{2}\right)\right).$$

Now for (4.17) and (4.18) we find

$$J_{1,k}=\int_k^{k+1}(\operatorname{sinc}(u)+\operatorname{sinc}(u-1)+\operatorname{sinc}(u+1)+\operatorname{sinc}(u))\,du,$$

$$J_{2,k}=\int_k^{k+1}(\operatorname{sinc}(u-1)+\operatorname{sinc}(u-2)+\operatorname{sinc}(u+2)+\operatorname{sinc}(u+1))\,du.$$

Let us prove that

$$S_k=\frac{1}{2}\int_k^{k+1}(\operatorname{sinc}(v+1)+\operatorname{sinc}(v))\,dv. \tag{4.19}$$

Indeed,

$$\begin{aligned}\int_k^{k+1}(\operatorname{sinc}(v+1)+\operatorname{sinc}(v))\,dv&=\int_k^{k+2}\operatorname{sinc}(v)dv=\int_0^{k+2}\operatorname{sinc}(v)dv-\int_0^{k}\operatorname{sinc}(v)dv\\&=\operatorname{Sci}(k+2)-\operatorname{Sci}(k).\end{aligned}$$

We have now by (4.19)

$$J_{1,k}=S_{k-1}+S_k,\ J_{2,k}=S_{k-2}+S_{k+1}. \tag{4.20}$$

Hence, by Lemma 2 we get the assertion. □

5. EXACT VALUES OF NORMS OF TRIGONOMETRIC BLACKMAN- AND ROGOSINSKI-TYPE OPERATORS

As we saw in Section 4, in the case $0\le a\le\frac{3}{8}$ the kernel function $s_{B,a}$ in (4.8) has sign changes only in integers. This fact allows us to calculate the norms comparatively easily.

Theorem 7. *If* $0\le a\le\frac{3}{8}$, *then*

$$\sup_n\|B_{n,a}\|_{[C_{2\pi}]}=2a(\operatorname{Sci}(2)+\operatorname{Sci}(3))+(1-2a)(\operatorname{Sci}(1)+\operatorname{Sci}(4)). \tag{5.1}$$

Proof. We have to calculate the signs of $s_{B,a}$ in (4.8). Since $p(t)\ge 0$, for signs of $s_{B,a}$ we find

$$\operatorname{sgn}(s_{B,a}(u))=\begin{cases}1, & 0\le u<3,\\(-1)^k, & k<u<k+1, k\ge 3.\end{cases}$$

Let us denote

$$I:=\int_0^\infty|s_{B,a}(u)|\,du. \tag{5.2}$$

Decomposing the integral (5.2), and keeping in mind the signs of $s_{B,a}$ and notation in (4.16), we obtain

$$I = \int_0^3 s_{B,a}(u)du + \sum_{k=3}^{\infty}(-1)^k \int_k^{k+1} s_{B,a}(u)du = I_{a,0} + I_{a,1} + I_{a,2} + \sum_{k=3}^{\infty}(-1)^k I_{a,k}. \tag{5.3}$$

We prove that the series

$$M_a := \sum_{k=3}^{\infty}(-1)^k I_{a,k} \tag{5.4}$$

converges absolutely. Indeed, since by (4.19)

$$S_k = \frac{1}{2}\int_k^{k+1} \frac{\sin \pi v dv}{\pi v(v+1)}, \tag{5.5}$$

we obtain $|S_k| \leq \frac{1}{2\pi}\frac{1}{k(k+1)} = O\left(\frac{1}{k^2}\right)$, hence by (4.16) we have $|I_{a,k}| = O_a\left(\frac{1}{k^2}\right)$, and consequently the series (5.4) converges absolutely. Thus, we may apply the scheme

$$\sum_{k=3}^{\infty} A_k = \sum_{k=1}^{\infty} A_{2k+1} + \sum_{k=1}^{\infty} A_{2k+2}$$

to the series (5.4), which gives

$$M_a = \sum_{k=3}^{\infty}(-1)^k I_{a,k} = \sum_{k=1}^{\infty}(I_{a,2k+2} - I_{a,2k+1}). \tag{5.6}$$

Using Lemma 3, we can write

$$I_{a,2k+2} - I_{a,2k+1} = \left(2a - \frac{1}{2}\right)(S_{2k+2} - S_{2k}) - \left(a - \frac{1}{2}\right)(S_{2k+3} - S_{2k-1}). \tag{5.7}$$

Considering the partial sums of M_a, we write

$$\begin{aligned} M_{a,N} :&= \left(2a - \frac{1}{2}\right)\sum_{k=1}^{N}(S_{2k+2} - S_{2k}) - \left(a - \frac{1}{2}\right)\sum_{k=1}^{N}(S_{2k+3} - S_{2k-1}) \\ &= \left(2a - \frac{1}{2}\right)(-S_2 + S_{2N+2}) - \left(a - \frac{1}{2}\right)(-S_1 - S_3 + S_{2N+3} + S_{2N+1}). \end{aligned}$$

Further, since by (5.5) $S_N = O(N^{-2})$, we get

$$M_a = \left(\frac{1}{2} - 2a\right)S_2 + \left(a - \frac{1}{2}\right)(S_1 + S_3) = -aS_2 + \left(\frac{1}{2} - a\right)(S_2 - S_1 - S_3).$$

Now by Lemma 3 and (5.4) for (5.3) we obtain

$$I = a(S_{-1} + 2S_0 + 2S_1) + \left(\frac{1}{2} - a\right)(S_{-2} + S_{-1} + S_0 + 2S_2). \tag{5.8}$$

By (4.15) we deduce (Sci is the odd function)

$$S_{-k} = S_{k-2}, \tag{5.9}$$

which for (5.8) yields

$$I = \left(\frac{S_{-1}}{2} + S_0 + S_2\right) + 2a(S_1 - S_2). \tag{5.10}$$

Finally, using (4.15), we find

$$I = \frac{1}{2}(\mathrm{Sci}(1) + \mathrm{Sci}(4)) + a(\mathrm{Sci}(3) - \mathrm{Sci}(1) - \mathrm{Sci}(4) + \mathrm{Sci}(2)). \tag{5.11}$$

Thus, by (4.7) and (5.2), the Blackman operator norm has the value

$$\sup_n \|B_{n,a}\|_{[C_{2\pi}]} = 2a(\mathrm{Sci}(2) + \mathrm{Sci}(3)) + (1-2a)(\mathrm{Sci}(1) + \mathrm{Sci}(4)).$$

□

Remark 8. If $0 \le a \le 3/8$, then by calculating the modified integral sine using Mathematica, we find

$$\sup_n \|B_{n,a}\|_{[C_{2\pi}]} = 1.06446... - a(0.15991...).$$

In the particular case $a = \frac{3}{8}$, we get

$$\sup_n \|B_{n,3/8}\|_{[C_{2\pi}]} = \frac{3}{4}\big(\mathrm{Sci}(2) + \mathrm{Sci}(3)\big) + \frac{1}{4}\big(\mathrm{Sci}(1) + \mathrm{Sci}(2)\big) = 1.00449...$$

That value is very close to the norms of positive operators which are equal to one. In a quite similar way we are able to calculate norms with some other values of parameter $a \in \mathbb{R}$. The results are summarized as follows.

Theorem 8. *The Blackman-type operators $B_{n,a} \in [C_{2\pi}]$ have the following operator norms*:
1. *If $a = 1/2$ (Hann operator, see Remark 5), then*

$$\sup_n \|B_{n,1/2}\|_{[C_{2\pi}]} = \mathrm{Sci}(2) + \mathrm{Sci}(1) = 1.0409...$$

2. *If $a = 27/64$ (then the polynomial (4.10) has an integer zero at $t = 3$), then*

$$\sup_n \|B_{n,27/64}\|_{[C_{2\pi}]} = \frac{1}{32}(5\,\mathrm{Sci}(2) + 27\,\mathrm{Sci}(3) + 27\,\mathrm{Sci}(4) + 5\,\mathrm{Sci}(5)) = 1.00235...$$

3. *If $a = 5/8$ ($B_{n,5/8}$ has the highest order of approximation, see Theorem 2), then*

$$\sup_n \|B_{n,5/8}\|_{[C_{2\pi}]} = 4\int_1^{\sqrt{5/2}} s_{B,5/8}(t)dt + \frac{1}{4}(12\,\mathrm{Sci}(1) - \mathrm{Sci}(2) - 6\,\mathrm{Sci}(3) + \mathrm{Sci}(4)) = 1.23423...$$

Analogously, we can calculate some exact operator norms of the Rogosinski-type operators. A selection of results follows.

Theorem 9. *The Rogosinski-type operators $R_{n,a} \in [C_{2\pi}]$, defined by the kernel function*

$$s_a^{\psi}(t) = \frac{2\cos(\pi t)[12a - 3 + (12-16a)t^2]}{\pi(1-4t^2)(9-4t^2)},$$

have the following operator norms:

1. *If* $0 \le a \le \frac{1}{4}$, *then* (t_a *is the positive zero of the polynomial* $p(t) = 12a - 3 + (12 - 16a)t^2$)

$$\sup_n \|R_{n,a}\| = -2\int_0^{t_a} s_a^{\Psi}(t)dt + 2\int_{t_a}^{1/2} s_a^{\Psi}(t)dt + (3-2a)\,\mathrm{Sci}(1) + (5a-3)\,\mathrm{Sci}(2) + (a-1)(\mathrm{Sci}(4) - 3\,\mathrm{Sci}(3)).$$

2. *If* $\frac{1}{4} \le a \le \frac{3}{4}$, *then* $(p(t) \ge 0)$

$$\sup_n \|R_{n,a}\| = 2\int_0^{1/2} s_a^{\Psi}(t)dt + (1-2a)\,\mathrm{Sci}(1) + (5a-3)\,\mathrm{Sci}(2) + (2a-2)\,\mathrm{Sci}(3).$$

Corollary 4. *The operator norms of the Rogosinski-type operators have the following numerical values*:
1. *If* $a = \frac{3}{4}$, *then* $\sup_n \|R_{n,\frac{3}{4}}\| = 1.88903...$
2. *If* $a = \frac{1}{2}$, *then* $\sup_n \|R_{n,\frac{1}{2}}\| = 1.39741...$

6. APPROXIMATION BY ROGOSINSKI OPERATORS OF THE FOURIER–CHEBYSHEV SERIES

Our next application of the general framework in Sections 1–3 leads to the Fourier–Chebyshev approximation, introduced and elaborated in paper [3] by P. L. Butzer and R. Stens. Now the Chebyshev cosine operator function (1.4) will be used in Definition 4. Similarly to the trigonometric case, the Rogosinski-type operators have the form

$$R^C_{n,a}(f,x) := \hat{f}_C(0) + 2\sum_{k=1}^{n} \Psi_a\left(\frac{k}{n+1}\right) \hat{f}_C(k)\, T_k(x), \tag{6.1}$$

where Ψ_a is defined by (4.4). General Theorem 3 yields

Theorem 10. *For every* $f \in C_{[-1,1]}$ *and all* $a \in \mathbb{R}$ *for the Rogosinski-type operators* $R^C_{n,a} : C_{[-1,1]} \to C_{[-1,1]}$ *it holds that*

$$\begin{aligned}\left\|R^C_{n,a} f - f\right\|_{C_{[-1,1]}} \le \Big(\left\|R^C_{n,a}\right\|_{[C_{[-1,1]}]} + |a| + |1-a|\Big)\, E_n(f) \\ + |a|\,\omega^C\left(f, \frac{\pi}{2(n+1)}\right) + |1-a|\,\omega^C\left(f, \frac{3\pi}{2(n+1)}\right).\end{aligned}$$

We have to notice here that the modulus of continuity

$$\omega^C(f,\delta) := \sup_{0 \le h \le \delta \le \pi} \left\|T^C_h f - f\right\|_{C_{[-1,1]}}$$

is well defined for every $f \in C_{[-1,1]}$ and $\omega^C(f,\delta)$ is a monotone decreasing function of $\delta \in [0,\pi]$ with $\lim_{\delta\to 0+} \omega^C(f,\delta) = 0$ (see [3]). The quantity $E_n(f)$ is the best algebraic approximation of $f \in C_{[-1,1]}$ by polynomials of degree not exceeding $n \in \mathbb{N}$. About operator norms $\left\|R^C_{n,a}\right\|_{[C_{[-1,1]}]}$, using a basic reference [3], we may explain the following. The operator (6.1) can be rewritten in integral form as ([3], Section 5)

$$R^C_{n,a}(f,x) = \frac{1}{\pi}\int_{-1}^{1} T^C_{\arccos u}(f,x)\, r^C_{n,a}(u) \frac{du}{\sqrt{1-u^2}}, \tag{6.2}$$

where the kernel function is given by

$$r_{n,a}^C(u) := 1 + 2\sum_{k=1}^{n} \psi_a\left(\frac{k}{n+1}\right) T_k(u). \tag{6.3}$$

It is known ([3], Lemma 1) that $\left\|T_h^C f\right\|_{C_{[-1,1]}} \leq \|f\|_{C_{[-1,1]}}$, i.e. $T = 1$ in Definition 1. Thus, by (6.2) we obtain

$$\left\|R_{n,a}^C\right\|_{[C_{[-1,1]}]} \leq \frac{1}{\pi}\int_{-1}^{1} \left|r_{n,a}^C(u)\right| \frac{du}{\sqrt{1-u^2}}.$$

By the substitution $u = \cos t$ here the right-hand side becomes

$$\frac{1}{\pi}\int_{0}^{\pi} \left|1 + 2\sum_{k=1}^{n} \psi_a\left(\frac{k}{n+1}\right)\cos kt\right| dt,$$

which appears to be ([2], Section 1.2.4) the operator norm $\|R_{n,a}\|_{[C_{2\pi}]}$ of (4.3). Therefore,

$$\left\|R_{n,a}^C\right\|_{[C_{[-1,1]}]} \leq \|R_{n,a}\|_{[C_{2\pi}]}. \tag{6.4}$$

In fact, inequality (6.4) is true for all Θ-means of the Fourier–Chebyshev series with the kernel function

$$c_n(u) := 1 + 2\sum_{k=1}^{\infty} \Theta_k(n)\cos ku,$$

provided $c_n \in L_{2\pi}^1$, $n \in \mathbb{N}$ ([2], Section 1.2.4).

In the particular case $a = 1$ the kernel function (6.3) can be written explicitly as

$$r_n^C(u) = \sin\frac{\pi}{2(n+1)} \times \frac{T_{n+1}(u)}{u - \cos\frac{\pi}{2(n+1)}},\ u \neq \cos\frac{\pi}{2(n+1)},$$

and $r_n^C\left(\cos\frac{\pi}{2n+2}\right) = n+1$. The three first kernel functions are $r_0^C(u) = 1$, $r_1^C(u) = \sqrt{2}u + 1$, and $r_2^C(u) = u(2u+\sqrt{3})$. The kernel function r_n^C is not positive, but it is technically simpler than that introduced in [3].

7. CONCLUSION

We introduced the Blackman- and Rogosinski-type approximation operators using the cosine operator function. This abstract setting is useful, because now we are able to consider different approximation problems from the unique point of view. Another feature of this paper is that in the trigonometric case we computed exact values of some operator norms of defined Blackman- and Rogosinski-type approximation operators.

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Blackmani ja Rogosinski tüüpi operaatoritega lähendamisest

Andi Kivinukk ja Anna Saksa

On defineeritud abstraktses Banachi ruumis Blackmani ja Rogosinski tüüpi operaatorid, kasutades koosinus-operaatori mõistet. Uus lähenemisviis võimaldab ühtsest seisukohast tõestada lähenduskiiruste hinnanguid ja rakendada saadud tulemusi trigonomeetrilistele Fourier' ridadele või ka Fourier'-Tšebõšovi ridadele.