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MATHEMATICS

On endomorphisms of groups of order 32 with maximal subgroups C_2^4 or C_4^2

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Abstract. It is proved that each group of order 32 that has a maximal subgroup isomorphic to $C_2 \times C_2 \times C_2 \times C_2$ or $C_4 \times C_4$ is determined by its endomorphism semigroup in the class of all groups.

Key words: group, semigroup, endomorphism semigroup.

1. INTRODUCTION

It is well known that all endomorphisms of an Abelian group form a ring and many of its properties can be characterized by this ring. An excellent overview of the present situation in the theory of endomorphism rings of groups is given by Krylov et al. [5]. All endomorphisms of an arbitrary group form only a semigroup. The theory of endomorphism semigroups of groups is quite modestly developed. In a number of our papers we have found some classes of groups that are determined by their endomorphism semigroups in the class of all groups. Note that if G is a fixed group and an isomorphism of semigroups $\mathrm{End}(G)$ and $\mathrm{End}(H)$, where H is an arbitrary group, always implies an isomorphism of G and H, then we say that the group G is determined by its endomorphism semigroup in the class of all groups. There exist also groups that are not determined by their endomorphism semigroup in the class of all groups.

We know a complete answer to this problem for finite groups of order less than 32. It was proved in [12] that among the finite groups of order less than 32 only the alternating group A_4 (also called the tetrahedral group) and the binary tetrahedral group $\langle a, b \mid b^3 = 1, aba = bab \rangle$ are not determined by their endomorphism semigroups in the class of all groups. These two groups are non-isomorphic, but their endomorphism semigroups are isomorphic. It was natural to consider the groups of order 32.

All groups of order 32 were described by Hall and Senior [4]. There exist exactly 51 non-isomorphic groups of order 32. In [4], these groups are numbered by $1, 2, \ldots, 51$. We shall mark these groups by $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_{51}$, respectively. The groups $\mathscr{G}_1 - \mathscr{G}_7$ are Abelian, and, therefore, are determined by their endomorphism semigroups in the class of all groups ([6], Theorem 4.2). In [2], it was proved that the groups of order 32 presentable in the form $(C_4 \times C_4) \rtimes C_2$ (C_k – the cyclic group of order k) are determined by their endomorphism semigroups in the class of all groups. The groups of this type are $\mathscr{G}_3, \mathscr{G}_{14}, \mathscr{G}_{16}, \mathscr{G}_{31}, \mathscr{G}_{34}, \mathscr{G}_{39}, \mathscr{G}_{41}$. In [3], it was proved that the groups of order 32 presentable in the form $(C_8 \times C_2) \rtimes C_2$ are determined by their endomorphism semigroups in the class of all groups. The groups of this type are $\mathscr{G}_4, \mathscr{G}_{17}, \mathscr{G}_{20}, \mathscr{G}_{26}, \mathscr{G}_{27}$. In [13], Theorem 1.1, it was proved that the groups of order 32 that

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have a maximal subgroup isomorphic to $C_4 \times C_2 \times C_2$ are determined by their endomorphism semigroup in the class of all groups. The groups of this type are $\mathscr{G}_2$–$\mathscr{G}_4$, $\mathscr{G}_8$–$\mathscr{G}_{14}$, $\mathscr{G}_{16}, \mathscr{G}_{18}, \mathscr{G}_{20}$, $\mathscr{G}_{36}$–$\mathscr{G}_{38}$. In [14], Theorem 1.1, it was proved that the groups of order 32 that have a maximal subgroup isomorphic to $C_8 \times C_2$ are determined by their endomorphism semigroup in the class of all groups. The groups of this type are

$$\mathscr{G}_4, \mathscr{G}_5, \mathscr{G}_6, \mathscr{G}_{17}, \mathscr{G}_{19}, \mathscr{G}_{20}, \mathscr{G}_{21}, \mathscr{G}_{22}, \mathscr{G}_{26}, \mathscr{G}_{27}, \mathscr{G}_{28}, \mathscr{G}_{29}, \mathscr{G}_{30}, \mathscr{G}_{32}.$$

In this paper, we consider the groups of order 32 that have a maximal subgroup isomorphic to $C_2^4 = C_2 \times C_2 \times C_2 \times C_2$ or $C_4^2 = C_4 \times C_4$ and prove the following theorem:

Theorem 1.1. *Each group of order 32 that has a maximal subgroup isomorphic to $C_2 \times C_2 \times C_2 \times C_2$ or $C_4 \times C_4$ is determined by its endomorphism semigroup in the class of all groups.*

The groups of order 32 that have a maximal subgroup isomorphic to $C_2 \times C_2 \times C_2 \times C_2$ or $C_4 \times C_4$ are:

$$\mathscr{G}_1 - \mathscr{G}_3, \mathscr{G}_5, \mathscr{G}_8, \mathscr{G}_{11}, \mathscr{G}_{14} - \mathscr{G}_{16}, \mathscr{G}_{19}, \mathscr{G}_{21}, \mathscr{G}_{31}, \mathscr{G}_{33} - \mathscr{G}_{35}, \mathscr{G}_{39} - \mathscr{G}_{41}.$$

To prove the theorem, the characterizations of these groups by their endomorphism semigroups will be given. These characterizations will be used in the proof of the theorem.

We shall use the following notations:
G – a group;
$\mathrm{End}(G)$ – the endomorphism semigroup of G;
C_k – the cyclic group of order k;
$Q_n = \langle a, b \mid a^{2^n} = 1,\ a^{2^{n-1}} = b^2,\ b^{-1}ab = a^{-1} \rangle$ – the generalized quaternion group ($n \geq 2$);
$Q = Q_2$ – the quaternion group;
$D_n = \langle a, b \mid b^2 = a^n = 1,\ b^{-1}ab = a^{-1} \rangle$ – the dihedral group of order $2n$;
$\mathbb{Z}_k$ – the ring of residual classes modulo k;
$\langle K, \ldots, g, \ldots \rangle$ – the subgroup generated by subsets $K, \ldots$ and elements $g, \ldots$;
$[a, b] = a^{-1}b^{-1}ab$ ($a, b \in G$);
G' – the commutator-group of G;
$\widehat{g}$ – the inner automorphism of G, generated by an element $g \in G$;
$o(g)$ – the order of an element $g \in G$;
$I(G)$ – the set of all idempotents of $\mathrm{End}(G)$;
$K(x) = \{z \in \mathrm{End}(G) \mid zx = xz = z\}$;
$J(x) = \{z \in \mathrm{End}(G) \mid zx = xz = 0\}$;
$H(x) = \{z \in \mathrm{End}(G) \mid xz = z,\ zx = 0\}$;
$C(x) = \{z \in \mathrm{End}(G) \mid zx = xz\}$;
$K(x)^*$ – the group of all invertible elements of the semigroup $K(x)$ with identity x, where $x \in I(G)$;
$V_{K(x)^*}(y) = \{z \in K(x)^* \mid zy = y\}$.

The set $K(x)$ is a subsemigroup of $\mathrm{End}(G)$. If $x \in I(G)$, then $K(x)$ is a monoid with unit x. We shall write the mapping to the right of the element on which it acts.

2. GROUPS THAT HAVE A MAXIMAL SUBGROUP $C_2 \times C_2 \times C_2 \times C_2$ OR $C_4 \times C_4$

In this section, using results obtained by Hall and Senior [4], the list of all groups of order 32 that have a maximal subgroup isomorphic to $C_2 \times C_2 \times C_2 \times C_2$ or $C_4 \times C_4$ is given. To this end, denote

$$\mathscr{G}_{16,1} = \langle a, b, c \mid a^4 = b^2 = c^2 = 1,\ ab = ba,\ bc = cb,\ c^{-1}ac = a^{-1}b \rangle.$$

The group $\mathscr{G}_{16,1}$ is a group of order 16. The groups of order 32 that have a maximal subgroup isomorphic to $C_2 \times C_2 \times C_2 \times C_2$ are

- $\mathscr{G}_1 = C_2 \times C_2 \times C_2 \times C_2 \times C_2$, $\mathscr{G}_2 = C_2 \times C_2 \times C_2 \times C_4$;
- $\mathscr{G}_8 = C_2 \times C_2 \times D_4$, $\mathscr{G}_{11} = C_2 \times \mathscr{G}_{16,1}$;
- $\mathscr{G}_{33} = \langle a, b, c, d, g \mid a^2 = b^2 = c^2 = d^2 = g^2 = 1,\ ab = ba,\ ac = ca,$
  $ad = da,\ bc = cb,\ bd = db,\ cd = dc,\ g^{-1}ag = b,\ g^{-1}cg = d \rangle.$

The groups of order 32 that have a maximal subgroup isomorphic to $C_4 \times C_4$ are

- $\mathscr{G}_3 = C_2 \times C_4 \times C_4$, $\mathscr{G}_5 = C_4 \times C_8$, $\mathscr{G}_{14} = C_4 \times D_4$, $\mathscr{G}_{15} = C_4 \times Q$;
- $\mathscr{G}_{16} = \langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ ab = ba,\ cb = bc,\ c^{-1}ac = ab^2 \rangle$
  $= (\langle a \rangle \times \langle b \rangle) \leftthreetimes \langle c \rangle = (C_4 \times C_4) \leftthreetimes C_2;$
- $\mathscr{G}_{19} = \langle a, b \mid a^4 = b^8 = 1,\ ab^2 = b^2a,\ b^{-1}ab = ab^4 \rangle$
  $= \langle a, b \mid a^4 = b^8 = 1,\ a^{-1}ba = b^5 \rangle = \langle b \rangle \leftthreetimes \langle a \rangle = C_8 \leftthreetimes C_4;$
- $\mathscr{G}_{21} = \langle a, b \mid a^4 = b^8 = 1,\ b^{-1}ab = a^{-1} \rangle = \langle a \rangle \leftthreetimes \langle b \rangle = C_4 \leftthreetimes C_8;$
- $\mathscr{G}_{31} = \langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ ab = ba,\ c^{-1}ac = b \rangle$
  $= (\langle a \rangle \times \langle b \rangle) \leftthreetimes \langle c \rangle = (C_4 \times C_4) \leftthreetimes C_2;$
- $\mathscr{G}_{34} = \langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ ab = ba,\ c^{-1}ac = a^{-1},$
  $c^{-1}bc = b^{-1} \rangle = (\langle a \rangle \times \langle b \rangle) \leftthreetimes \langle c \rangle = (C_4 \times C_4) \leftthreetimes C_2;$
- $\mathscr{G}_{35} = \langle a, b, c \mid a^4 = b^4 = 1,\ ab = ba,\ c^2 = a^2,\ c^{-1}ac = a^{-1},$
  $c^{-1}bc = b^{-1} \rangle = \langle b \rangle \leftthreetimes \langle a, c \rangle = C_4 \leftthreetimes Q;$
- $\mathscr{G}_{39} = \langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ ab = ba,\ c^{-1}ac = ab^2,$
  $c^{-1}bc = ba^2 \rangle = (\langle a \rangle \times \langle b \rangle) \leftthreetimes \langle c \rangle = (C_4 \times C_4) \leftthreetimes C_2;$
- $\mathscr{G}_{40} = \langle a, b, c \mid a^4 = b^4 = 1,\ ab = ba,\ c^2 = a^2b^2,\ c^{-1}ac = a^{-1},$
  $c^{-1}bc = b^{-1}a^2 \rangle;$
- $\mathscr{G}_{41} = \langle a, b, c \mid a^4 = b^4 = c^2 = 1,\ ab = ba,\ c^{-1}ac = a^{-1}b^2,$
  $c^{-1}bc = ba^2 \rangle = (\langle a \rangle \times \langle b \rangle) \leftthreetimes \langle c \rangle = (C_4 \times C_4) \leftthreetimes C_2.$

It is known that the following groups are determined by their endomorphism semigroups in the class of all groups: finite Abelian groups ([6], Theorem 4.2), dihedral 2-groups ([8], Theorem 3.1), generalized quaternion groups ([9]), finite groups of order 16 ([11]). On the other hand, if the groups $G_1, G_2, \ldots, G_n$ are determined by their endomorphism semigroups in the class of all groups, then so is their direct product $G_1 \times G_2 \times \ldots \times G_n$ ([6], Theorem 1.13). Therefore, the groups $\mathscr{G}_1 - \mathscr{G}_3$, $\mathscr{G}_5$, $\mathscr{G}_8$, $\mathscr{G}_{11}$, $\mathscr{G}_{14}$, and $\mathscr{G}_{15}$ are determined by their endomorphism semigroups in the class of all groups. In [2], it was proved that the groups of order 32 presentable in the form $(C_4 \times C_4) \leftthreetimes C_2$ are determined by their endomorphism semigroups in the class of all groups. Therefore, the groups $\mathscr{G}_{16}$, $\mathscr{G}_{31}$, $\mathscr{G}_{34}$, $\mathscr{G}_{39}$, and $\mathscr{G}_{41}$ are determined by their endomorphism semigroups in the class of all groups. In [10], Theorem, it was proved that the semidirect product $G = C_{p^n} \leftthreetimes C_m$, where p is a prime number, n and m are some positive integers, is determined by its endomorphism semigroup in the class of all groups. Hence the groups $\mathscr{G}_{19}$ and $\mathscr{G}_{21}$ are determined by their endomorphism semigroups in the class of all groups. To prove Theorem 1.1, we have to prove in addition that the groups $\mathscr{G}_{33}$, $\mathscr{G}_{35}$, and $\mathscr{G}_{40}$ are determined by their endomorphism semigroups in the class of all groups. It is done in theorems 4.2, 5.2, and 6.2.

3. PRELIMINARY LEMMAS

For convenience of reference, let us recall some known facts that will be used in the proofs of our main results.

Lemma 3.1. *If $x \in I(G)$, then $G = \mathrm{Ker}\,x \leftthreetimes \mathrm{Im}\,x$ and $\mathrm{Im}\,x = \{g \in G \mid gx = g\}$.*

Lemma 3.2. *If $x \in I(G)$, then*

$$K(x) = \{\, y \in \mathrm{End}(G) \mid (\mathrm{Im}\,x)y \subset \mathrm{Im}\,x,\ (\mathrm{Ker}\,x)y = \langle 1 \rangle \,\}$$

and $K(x)$ is a subsemigroup with the unity x of $\mathrm{End}(G)$ that is canonically isomorphic to $\mathrm{End}(\mathrm{Im}\,x)$. In this isomorphism element y of $K(x)$ corresponds to its restriction on the subgroup $\mathrm{Im}\,x$ of G.

Lemma 3.3. *If $x \in I(G)$, then*

$$J(x) = \{z \in \mathrm{End}(G) \mid (\mathrm{Im}x)z = \langle 1 \rangle,\ (\mathrm{Ker}x)z \subset \mathrm{Ker}x\}.$$

Lemma 3.4. *If $x \in I(G)$, then*

$$H(x) = \{\, y \in \mathrm{End}(G) \mid (\mathrm{Im}x)y \subset \mathrm{Ker}x,\ (\mathrm{Ker}x)y = \langle 1 \rangle \,\}.$$

Lemma 3.5. *If $x \in I(G)$, then*

$$P(x) = \{\, y \in \mathrm{End}(G) \mid y|\mathrm{Im}x = 1_{\mathrm{Im}x},\ (\mathrm{Ker}x)y \subset \mathrm{Ker}x \,\}.$$

We omit the proofs of these lemmas because these are straightforward corollaries from the definitions.

Lemma 3.6. *If*

$$D_4 = \langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ ab = ba,\ c^{-1}ac = b\rangle \tag{3.1}$$

and $x \in I(D_4)$ such that $\mathrm{Im}x = \langle c \rangle$, $\mathrm{Ker}x = \langle a \rangle \times \langle b \rangle$, then 1^0 $K(x) \cong \mathrm{End}(C_2)$ and 2^0 $|\{u \in \mathrm{End}(D_4) \mid xu = u,\ ux = 0\}| = 4$. Conversely, if $x \in I(D_4)$ satisfies 1^0 and 2^0, then there exist $a, b, c \in D_4$ such that

$$D_4 = \mathrm{Ker}x \leftthreetimes \mathrm{Im}x,\ \mathrm{Im}x = \langle c \rangle \cong C_2,\ \mathrm{Ker}x = \langle a \rangle \times \langle b \rangle \cong C_2 \times C_2$$

and (3.1) *holds.*

Lemma 3.6 is obtained by easy calculations in the group D_4.

Lemma 3.7. ([9], Theorems 2.1 and 3.1) *Assume that a group G decomposes into a semidirect product*

$$G = H \leftthreetimes ((G_1 \times \ldots \times G_n) \leftthreetimes K),\ n \geq 2, \tag{3.2}$$

where

$$\langle G_i, K \rangle = G_i \leftthreetimes K \ \ (i = 1, 2, \ldots, n). \tag{3.3}$$

Denote by x and x_i the projections of G onto K and $G_i \leftthreetimes K$ $(i = 1, 2, \ldots, n)$, i.e.

$$\mathrm{Im}x_i = G_i \leftthreetimes K,\ \ \mathrm{Ker}x_i = H \leftthreetimes \prod_{j=1,\, j\neq i}^{n} G_j, \tag{3.4}$$

$$\mathrm{Im}x = K,\ \ \mathrm{Ker}x = H \leftthreetimes (G_1 \times \ldots \times G_n), \tag{3.5}$$

$$G_i = \mathrm{Ker}x \cap \mathrm{Im}x_i,\ \ H = \cap_{j=1}^{n} \mathrm{Ker}x_j. \tag{3.6}$$

Then

$$x_i x_j = x_j x_i = x;\ i,\ j = 0, 1, \ldots, n,\ i \neq j, \tag{3.7}$$

and for each $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, there exists $z_{ij} = z_{ji} \in I(G)$ that satisfies the following properties:
1^0 $x_i, x_j \in K(z_{ij})$,
2^0 *there exists a unique pair V_i, V_j of subgroups of $K(z_{ij})^*$ with properties*

(i) $V_i \subset C(x_i)$, $V_j \subset C(x_j)$;

(ii) $V_i x_i = V_{K(x_i)^*}(x)$, $V_j x_j = V_{K(x_j)^*}(x)$;

(iii) $x_i v x_i = x_i$ *for each* $v \in V_j$;

(iv) $x_j u x_j = x_j$ *for each* $u \in V_i$.

Conversely, suppose that there exist idempotents x, x_1,..., x_n of $\mathrm{End}(G)$ *such that* (3.7) *holds and for each $i, j \in \{1, 2,..., n\}$, $i \neq j$, there exists $z_{ij} = z_{ji} \in I(G)$ that satisfies the properties 1^0 and 2^0. Then the group G decomposes into the semidirect product* (3.2), *where the equalities* (3.3)–(3.6) *are true. In this case the set $B = \{y \in I(G) \mid x_1,..., x_n \in K(y)\}$ is non-empty and there exists a unique $z \in B$ such that $zy = yz = z$ for each $y \in B$. This z is the projection of G onto its subgroup $(G_1 \times \ldots \times G_n) \leftthreetimes K$ and* $\mathrm{Ker}z = H$.

Denote by $\mathscr{C}(x; x_1, \ldots, x_n)$ the set of the conditions for $x; x_1, \ldots, x_n$ given in the second part of Lemma 3.7 (i.e. equalities (3.7) and 1^0, 2^0). If the condition $\mathscr{C}(x; x_1, \ldots, x_n)$ is satisfied, denote by $\pi_{\mathscr{C}}$ the projection of G onto its subgroup $(G_1 \times \ldots \times G_n) \leftthreetimes K$. The endomorphism $\pi_{\mathscr{C}}$ is a unique $z \in B$ such that $zy = yz = z$ for each $y \in B$. Denote by $\mathscr{C}_0(x; x_1, \ldots, x_n)$ the condition $\mathscr{C}(x; x_1, \ldots, x_n)$ with $\pi_{\mathscr{C}} = 1_G$ (i.e. $H = \langle 1 \rangle$).

4. GROUP $\mathscr{G}_{33}$

In this section, we shall characterize the group

$$\mathscr{G}_{33} = \langle a, b, c, d, g \mid a^2 = b^2 = c^2 = d^2 = g^2 = 1,\ ab = ba,\ ac = ca,$$
$$ad = da,\ bc = cb, bd = db,\ cd = dc,\ g^{-1}ag = b,\ g^{-1}cg = d \rangle$$

by its endomorphism semigroup. The group $\mathscr{G}_{33}$ is a group of order 32 and the numbers of its elements of orders 2 and 4 are 19 and 12, respectively ([4]). Clearly,

$$\begin{aligned}
\mathscr{G}_{33} &= (\langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle) \leftthreetimes \langle g \rangle \cong (C_2 \times C_2 \times C_2 \times C_2) \leftthreetimes C_2; \\
\mathscr{G}_{33} &= (\langle c \rangle \times \langle d \rangle) \leftthreetimes ((\langle a \rangle \times \langle b \rangle) \leftthreetimes \langle g \rangle);\ (\langle a \rangle \times \langle b \rangle) \leftthreetimes \langle g \rangle \cong D_4; \\
\mathscr{G}_{33} &= (\langle a \rangle \times \langle b \rangle) \leftthreetimes ((\langle c \rangle \times \langle d \rangle) \leftthreetimes \langle g \rangle),\ (\langle c \rangle \times \langle d \rangle) \leftthreetimes \langle g \rangle \cong D_4.
\end{aligned}$$

Our aim is to prove the following theorem.

Theorem 4.1. *A finite group G is isomorphic to $\mathscr{G}_{33}$ if and only if there exist $x, y, z \in I(G)$ that satisfy condition $\mathscr{C}_0(x; y, z)$ and the following properties*:
1^0 $K(x) \cong \mathrm{End}(C_2)$;
2^0 $K(y) \cong K(z) \cong \mathrm{End}(D_4)$;
3^0 $|\{u \in K(y) \mid xu = u,\ ux = 0\}| = |\{u \in K(z) \mid xu = u,\ ux = 0\}| = 4.$

Proof. Necessity. Let $G = \mathscr{G}_{33}$. We have to prove that there exist $x, y, z \in I(G)$ that satisfy condition $\mathscr{C}_0(x; y, z)$ and properties 1^0–3^0 of the theorem.

Denote
$$K = \langle g \rangle,\ G_1 = \langle a \rangle \times \langle b \rangle,\ G_2 = \langle c \rangle \times \langle d \rangle,\ H = \langle 1 \rangle.$$
Since $\langle a, b, g \rangle \cong \langle c, d, g \rangle \cong D_4$, we have

$$G = H \leftthreetimes ((G_1 \times G_2) \leftthreetimes K),$$

$$\langle G_1, K \rangle = G_1 \leftthreetimes K \cong D_4,\ \langle G_2, K \rangle = G_2 \leftthreetimes K \cong D_4.$$

We can use now Lemma 3.7 for the case $n = 2$. By Lemma 3.7, there exist $x, y, z \in I(G)$ $(x_1 = y, x_2 = z)$ that satisfy condition $\mathscr{C}_0(x; y, z)$:

$$\mathrm{Im}\, x = K \cong C_2,\ \mathrm{Im}\, y = G_1 \leftthreetimes K \cong D_4,\ \mathrm{Im}\, z = G_2 \leftthreetimes K \cong D_4.$$

Lemma 3.2 implies that

$$K(x) \cong \mathrm{End}(C_2),\ K(y) \cong \mathrm{End}(D_4),\ K(z) \cong \mathrm{End}(D_4).$$

Hence properties 1^0 and 2^0 of the theorem hold. In view of Lemma 3.6 (use it for the groups $\mathrm{Im}\, y$ and $\mathrm{Im}\, z$), property 3^0 is also true. The necessity is proved.

Sufficiency. Let G be a finite group and there exist $x, y, z \in I(G)$ that satisfy property $\mathscr{C}_0(x; y, z)$ and properties 1^0–3^0 of the theorem. Our aim is to prove that $G \cong \mathscr{G}_{33}$.

By Lemma 3.7, G decomposes into the semidirect product

$$G = (G_1 \times G_2) \leftthreetimes K,$$

where

$$\mathrm{Im}\,x = K,\ \mathrm{Im}\,y = G_1 \leftthreetimes K,\ \mathrm{Im}\,z = G_2 \leftthreetimes K,$$

$$\mathrm{Ker}\,x = G_1 \times G_2,\ \mathrm{Ker}\,y = G_2,\ \mathrm{Ker}\,z = G_1.$$

In view of Lemma 3.2 and properties 1^0 and 2^0,

$$\mathrm{End}(K) = \mathrm{End}(\mathrm{Im}\,x) \cong \mathrm{End}(C_2),$$

$$\mathrm{End}(G_1 \leftthreetimes K) = \mathrm{End}(\mathrm{Im}\,y) \cong \mathrm{End}(D_4),$$

$$\mathrm{End}(G_2 \leftthreetimes K) = \mathrm{End}(\mathrm{Im}\,z) \cong \mathrm{End}(D_4).$$

Since each finite Abelian group and dihedral 2-groups are determined by their endomorphism semigroups in the class of all groups ([6], Theorem 4.2 and [8], Theorem 3.1), we have

$$K = \langle g \rangle \cong C_2,\ G_1 \leftthreetimes K \cong G_2 \leftthreetimes K \cong D_4$$

for some $g \in G$.

Let us use Lemma 3.6 for the groups $\mathrm{Im}\,y \cong D_4$ and $\mathrm{Im}\,z \cong D_4$. In view of $\mathrm{Im}\,x \cong C_2$ and property 3^0, conditions 1^0 and 2^0 of Lemma 3.6 are satisfied. Therefore, there exist $a, b \in G_1 = \mathrm{Im}\,y \cap \mathrm{Ker}\,x$ and $c, d \in \mathrm{Im}\,z \cap \mathrm{Ker}\,x$ such that

$$\mathrm{Im}\,y = (\langle a \rangle \times \langle b \rangle) \leftthreetimes \langle g \rangle = \langle a, b, g \mid a^2 = b^2 = g^2 = 1, ab = ba, g^{-1}ag = b \rangle,$$

$$\mathrm{Im}\,z = (\langle c \rangle \times \langle d \rangle) \leftthreetimes \langle g \rangle = \langle c, d, g \mid c^2 = d^2 = g^2 = 1, cd = dc, g^{-1}cg = d \rangle.$$

We have proved that

$$G = \langle a, b, c, d, g \mid a^2 = b^2 = c^2 = d^2 = g^2 = 1,\ ab = ba,\ ac = ca,$$

$$ad = da,\ bc = cb, bd = db,\ cd = dc,\ g^{-1}ag = b,\ g^{-1}cg = d \rangle,$$

i.e. $G \cong \mathscr{G}_{33}$. The sufficiency is proved.

The theorem is proved.

Theorem 4.2. *The group $\mathscr{G}_{33}$ is determined by its endomorphism semigroup in the class of all groups.*

Proof. Let G^* be a group such that the endomorphism semigroups of G^* and $\mathscr{G}_{33}$ are isomorphic:

$$\mathrm{End}(G^*) \cong \mathrm{End}(\mathscr{G}_{33}). \tag{4.1}$$

Denote by z^* the image of $z \in \mathrm{End}(\mathscr{G}_{33})$ in isomorphism (4.1). Since $\mathrm{End}(G^*)$ is finite, so is G^* ([1], Theorem 2). By Theorem 4.1, there exist $x, y, z \in I(\mathscr{G}_{33})$ that satisfy condition $\mathscr{C}_0(x; y, z)$ and properties 1^0–3^0 of the theorem. In view of isomorphism (4.1), the endomorphisms x^*, y^*, z^* satisfy condition $\mathscr{C}_0(x^*; y^*, z^*)$ and properties 1^0–3^0, where x, y, and z are always replaced by x^*, y^*, and z^*, respectively. Using now Theorem 4.1 for G^*, it follows that G^* and $\mathscr{G}_{33}$ are isomorphic. The theorem is proved.

5. GROUP $\mathscr{G}_{35}$

In this section, we shall characterize the group

$$\mathscr{G}_{35} = \langle a, b, c \mid a^4 = b^4 = 1,\ ab = ba,\ c^2 = a^2,\ c^{-1}ac = a^{-1},\ c^{-1}bc = b^{-1}\rangle$$

by its endomorphism semigroup. The group $\mathscr{G}_{35}$ is a group of order 32 and the numbers of its elements of orders 2 and 4 are 3 and 28, respectively ([4]). Clearly,

$$\mathscr{G}_{35} = \langle b\rangle \rtimes \langle a, c\rangle = \langle b\rangle \rtimes Q,\ \ Q = \langle a, c\rangle.$$

It is easy to check that

$$\mathscr{G}_{35}' = Z(\mathscr{G}_{35}) = \langle a^2\rangle \times \langle b^2\rangle \cong C_2 \times C_2,$$

$$\mathscr{G}_{35}/\mathscr{G}_{35}' = \langle a\mathscr{G}_{35}'\rangle \times \langle b\mathscr{G}_{35}'\rangle \times \langle c\mathscr{G}_{35}'\rangle \cong C_2 \times C_2 \times C_2.$$

Each element of $\mathscr{G}_{35}$ can be presented in the canonical form $c^i a^j b^k$, where $i \in \{0, 1\},\ j, k \in \mathbb{Z}_4$.

Our aim is to prove the following theorem.

Theorem 5.1. *A finite group G is isomorphic to $\mathscr{G}_{35}$ if and only if* $\mathrm{Aut}(G)$ *is a* 2*-group and there exists $x \in I(G)$ such that the following properties hold*:
1^0 $\quad K(x) \cong \mathrm{End}(Q)$;
2^0 $\quad J(x) \cap I(G) = \{0\}$;
3^0 $\quad |H(x)| = 4$;
4^0 $\quad P(x) \cong \mathrm{End}(C_4)$.

Proof. Necessity. Let $G = \mathscr{G}_{35}$. It is known that $|\mathrm{Aut}(G)| = 2^9$, i.e. $\mathrm{Aut}(G)$ is a 2-group ([4]). Denote by x the projection of G onto its subgroup $Q = \langle a, c\rangle$. Then $\mathrm{Im}\,x = Q$ and $\mathrm{Ker}\,x = \langle b\rangle$. By Lemma 3.2, $K(x) \cong \mathrm{End}(\mathrm{Im}\,x) \cong \mathrm{End}(Q)$, i.e. property 1^0 holds.

Note that each endomorphism of G is uniquely determined by its images on the generators a, b, and c. By Lemma 3.3, $z \in J(x)$ has the form

$$cz = az = 1,\ bz = b^i,\ i \in \mathbb{Z}_4. \tag{5.1}$$

The map $z : G \longrightarrow G$ given by (5.1) preserves the generating relations of G and induces an endomorphism of G if and only if $(bz)^2 = 1$, i.e. $i \equiv 0 \,(\mathrm{mod}\, 2)$. Hence

$$J(x) = \{z \mid az = cz = 1,\ bz = b^{2i_0};\ i_0 \in \mathbb{Z}_2\}.$$

Since $bz^2 = b^{4i_0^2} = 1$, we have $z^2 = z$ if and only if $i_0 = 0$, i.e. $z = 0$. Therefore, $J(x) \cap I(G) = \{0\}$ and property 2^0 holds.

By Lemma 3.4, $H(x)$ consists of endomorphisms $z : G \longrightarrow G$ such that

$$az = b^i,\ cz = b^j,\ bz = 1 \tag{5.2}$$

for some $i, j \in \mathbb{Z}_4$. The map z given by (5.2) preserves the generating relations of G and induces an endomorphism of G if and only if $b^{2i} = b^{2j} = 1$, i.e. $i \equiv j \equiv 0 \,(\mathrm{mod}\, 2)$. Hence

$$H(x) = \{z \mid az = b^{2i_0},\ cz = b^{2j_0},\ bz = 1;\ i_0, j_0 \in \mathbb{Z}_2\}$$

and $|H(x)| = 4$. Property 3^0 is satisfied.

By Lemma 3.5, $P(x)$ consists of endomorphisms $z : G \longrightarrow G$ such that

$$az = a,\ cz = c,\ bz = b^i \tag{5.3}$$

for some $i \in \mathbb{Z}_4$. The map z given by (5.3) preserves the generating relations of G and induces an endomorphism for each $i \in \mathbb{Z}_4$. Therefore, $P(x) \cong \mathrm{End}(C_4)$ and property 4^0 is true. The necessity is proved.

Sufficiency. Let G be a finite group such that $\mathrm{Aut}(G)$ is a 2-group and let there exist $x \in I(G)$ that satisfies properties 1^0–4^0 of the theorem. Our aim is to prove that $G \cong \mathscr{G}_{35}$.

Since $x \in I(G)$, Lemma 3.1 implies that

$$G = \mathrm{Ker}x \leftthreetimes \mathrm{Im}x.$$

By property 1^0,

$$\mathrm{End}(\mathrm{Im}x) \cong \mathrm{End}(Q).$$

Since the quaternion group Q is determined by its endomorphism semigroup in the class of all groups ([7], Corollary 1), there exist $a, b \in G$ such that

$$\mathrm{Im}x = \langle a, c \mid a^4 = 1,\ c^2 = a^2,\ c^{-1}ac = a^{-1} \rangle \cong Q,$$

$$G = \mathrm{Ker}x \leftthreetimes Q = \mathrm{Ker}x \leftthreetimes \langle a, c \rangle.$$

Since $\mathrm{Aut}(G)$ is a 2-group, we have $\widehat{g} = 1$ for each $2'$-element $g \in G$. Therefore, each $2'$-element of G belongs into the centre of G. On the other hand, each $2'$-element of G belongs into $\mathrm{Ker}x$. Hence G splits into the direct product $G = G_2 \times G_{2'}$ of its Sylow 2-subgroup G_2 and Hall $2'$-subgroup $G_{2'}$. Clearly, $a, c \in G_2$ and $G_{2'} \subset \mathrm{Ker}x$. Denote by z the projection of G onto its subgroup $G_{2'}$. Then $z \in J(x) \cap I(G)$. By property 2^0, $z = 0$, i.e. $G_{2'} = \langle 1 \rangle$ and G is a 2-group. Clearly, $\mathrm{Ker}x \neq \langle 1 \rangle$.

Denote $M = \langle \mathrm{Ker}x, a^2 \rangle$. Then

$$G/M = \langle aM \rangle \times \langle cM \rangle \cong C_2 \times C_2.$$

Choose an element $d \in \mathrm{Ker}x$ of order two and consider the endomorphisms $z_{ij} = \varepsilon\pi_{ij}$ of G:

$$z_{ij} = \varepsilon\pi_{ij} : G \xrightarrow{\varepsilon} G/M \xrightarrow{\pi_{ij}} \langle d \rangle,$$

where ε is the natural homomorphism and

$$(aM)z_{ij} = d^i,\ (cM)z_{ij} = d^j;\ i, j \in \mathbb{Z}_2.$$

The number of such endomorphisms is four and all of them belong to $H(x)$. By property 3^0,

$$H(x) = \{z_{ij} \mid i, j \in \mathbb{Z}_2\}. \tag{5.4}$$

Therefore, $\mathrm{Ker}x$ has only one element of order two. By [15], Theorem 5.3.6, $\mathrm{Ker}x$ is cyclic or a generalized quaternion group Q_n ($n \geq 2$).

Assume that

$$\mathrm{Ker}x = Q_n = \langle a_1, b_1 \mid a_1^{2^n} = 1,\ a_1^{2^{n-1}} = b_1^2,\ b_1^{-1}a_1b_1 = a_1^{-1} \rangle.$$

The map

$$\tau:\ a \longmapsto a_1^{2^{n-2}},\ c \longmapsto b_1$$

can be extended to a homomorphism $\tau : \mathrm{Im}x = Q = \langle a, c \rangle \longrightarrow Q_n = \mathrm{Ker}x$ and we get the endomorphism $z = x\tau$ of G. Then $z \in H(x)$ and $z \neq z_{ij}$ for each $i, j \in \mathbb{Z}_2$. This contradicts (5.4) and, therefore, $\mathrm{Ker}x$ can not be a generalized quaternion group. Hence $\mathrm{Ker}x$ is cyclic:

$$\mathrm{Ker}x = \langle b \rangle \cong C_{2^n},\ n \geq 1,$$

$$G = \langle a, b, c \rangle = \langle b \rangle \leftthreetimes \langle a, c \rangle,$$

$$a^{-1}ba = b^r,\ c^{-1}bc = b^\rho$$

for some $b \in \mathrm{Ker}\,x$ and $r, \rho \in \mathbb{Z}^*_{2^n}$.

By Lemma 3.5, the subsemigroup $P(x)$ of $\mathrm{End}(G)$ consists of endomorphisms z of G, which can be presented on the generators as follows:

$$az = a,\ cz = c,\ bz = b^k,\ \ k \in \mathbb{Z}_{2^n}. \tag{5.5}$$

The map z given by (5.5) preserves the generating relations of G and induces an endomorphism of G for each $k \in \mathbb{Z}_{2^n}$. It follows from here that $P(x) \cong \mathrm{End}(C_{2^n})$. By property 4^0, $n = 2$, i.e. $\langle b \rangle \cong C_4$ and

$$r = \pm 1,\ \rho = \pm 1.$$

Let us consider all possible cases for r and ρ.

The case $r = \rho = 1$ is impossible, because then $G = \langle b \rangle \times \langle a, c \rangle$ and the projection of G onto its subgroup $\langle b \rangle$ is a non-zero element of $J(x) \cap I(G)$, which contradicts property 2^0.

Assume that $r = \rho = -1$, i.e.

$$a^{-1}ba = b^{-1},\ c^{-1}bc = b^{-1}.$$

Denote $a_1 = ca^{-1}$. Then

$$\begin{aligned}
a_1^2 &= ca^{-1} \cdot ca^{-1} = c^2 \cdot c^{-1}a^{-1}c \cdot a^{-1} = c^2 \cdot a \cdot a^{-1} = c^2,\\
a_1^4 &= 1,\\
c^{-1}a_1c &= c^{-1}ca^{-1}c = a^{-1}c = a^3c = a \cdot a^2c\\
&= a \cdot c^2c = ac^{-1} = (ca^{-1})^{-1} = a_1^{-1},\\
a^{-2}ba^2 &= a^{-1}(a^{-1}ba)a = a^{-1}b^{-1}a = b,\ a^2b = ba^2,\\
ba_1 &= bca^{-1} = c \cdot c^{-1}bc \cdot a^{-1} = cb^{-1}a^{-1} = ca \cdot a^{-1}b^{-1}a \cdot a^{-2}\\
&= caba^{-2} = caba^2 = caa^2b = ca^{-1}b = a_1b.
\end{aligned}$$

Therefore, the map

$$a \longmapsto a_1,\ c \longmapsto c,\ b \longmapsto b$$

induces an isomorphism $\mathscr{G}_{35} \cong G$, and the statement of the sufficiency is true.

If $r = 1,\ \rho = -1$, then $\mathscr{G}_{35} \cong G$, because the groups $\mathscr{G}_{35}$ and G have the same generating relations. If $r = -1,\ \rho = 1$, then the map

$$a \longmapsto c,\ c \longmapsto a,\ b \longmapsto b$$

induces an isomorphism $\mathscr{G}_{35} \cong G$. The sufficiency is proved.

The theorem is proved.

Theorem 5.2. *The group $\mathscr{G}_{35}$ is determined by its endomorphism semigroup in the class of all groups.*

The proof of Theorem 5.2 is similar to that of Theorem 4.2.

6. GROUP $\mathscr{G}_{40}$

In this section, we shall characterize the group

$$\mathscr{G}_{40} = \langle a, b, c \mid a^4 = b^4 = 1,\ ab = ba,\ c^2 = a^2b^2,\ \begin{cases} c^{-1}ac = a^{-1} \\ c^{-1}bc = b^{-1}a^2 \end{cases} \rangle$$

$$= \langle a, b, c \mid a^4 = b^4 = 1,\ ab = ba,\ c^2 = a^2b^2,\ c^{-1}ac = a^{-1},\ b^{-1}cb = c^{-1}\rangle$$

by its endomorphism semigroup. The group $\mathscr{G}_{40}$ is a group of order 32 and the numbers of its elements of orders 2 and 4 are 3 and 28, respectively ([4]). It is easy to check that

$$\mathscr{G}_{40}' = Z(\mathscr{G}_{40}) = \langle a^2\rangle \times \langle b^2\rangle \cong C_2 \times C_2,$$

$$\mathscr{G}_{40}/\mathscr{G}_{40}' = \langle a\mathscr{G}_{40}'\rangle \times \langle b\mathscr{G}_{40}'\rangle \times \langle c\mathscr{G}_{40}'\rangle \cong C_2 \times C_2 \times C_2.$$

Each element of $\mathscr{G}_{40}$ can be presented in the canonical form $c^ia^jb^k$, where $i \in \{0, 1\},\ j, k \in \mathbb{Z}_4$.

Our aim is to prove the following theorem.

Theorem 6.1. *A finite group G is isomorphic to $\mathscr{G}_{40}$ if and only if the following properties hold*:
1^0 $|\mathrm{Aut}(G)| = 2^8 = 256$;
2^0 $|\mathrm{End}(G) \setminus \mathrm{Aut}(G)| = 2^6 = 64$;
3^0 $x, y \in \mathrm{End}(G) \setminus \mathrm{Aut}(G) \implies xy = 0$;
4^0 *if $z \in \mathrm{Aut}(G)$ and $zy = y$ for each $y \in \mathrm{End}(G) \setminus \mathrm{Aut}(G)$, then $z^2 = 1$.*

Proof. Necessity. Let $G = \mathscr{G}_{40}$. To prove properties 1^0–4^0 for G, we have to find the endomorphisms of G. An endomorphism of G is fully determined by its action on the generators c, b, and a. Choose $z \in \mathrm{End}(G)$. Then

$$az = c^ia^jb^k,\ bz = c^la^mb^n,\ cz = c^sa^tb^u, \tag{6.1}$$

where

$$i, l, s \in \{0, 1\};\ j, k, m, n, t, u \in \mathbb{Z}_4.$$

The map z given by (6.1) induces an endomorphism of G if and only if it preserves the defining relations of G. After easy calculations, we obtain:
(1) the proper endomorphisms of G are the maps z, where

$$az = a^{2j}b^{2k},\ bz = a^{2m}b^{2n},\ cz = a^{2t}b^{2u};\ j, k, m, n, t, u \in \mathbb{Z}_2; \tag{6.2}$$

(2) the automorphisms of G are the maps z, where

$$\begin{cases} az = a^jb^k,\ bz = a^mb^n,\ cz = ca^tb^u; \\ j, k, m, n, t, u \in \mathbb{Z}_4, \\ j \equiv n \equiv 1 \,(\mathrm{mod}\,2),\ k \equiv 0 \,(\mathrm{mod}\,2),\ m \equiv u \,(\mathrm{mod}\,2). \end{cases} \tag{6.3}$$

Hence

$$|\mathrm{Aut}(G)| = 2^8 = 256,\ |\mathrm{End}(G) \setminus \mathrm{Aut}(G)| = 2^6 = 64$$

and properties 1^0 and 2^0 hold.

By (6.2), $\mathrm{Im}\,z \subset \langle a^2\rangle \times \langle b^2\rangle \subset \mathrm{Ker}\,z$ for all proper endomorphism z of G. Hence $\mathrm{Im}\,x \subset \mathrm{Ker}\,y$ for all proper endomorphisms x and y of G, and, therefore, $xy = 0$. Property 3^0 is true.

To prove property 4^0, choose an automorphism z given by (6.3) and a proper endomorphism y:

$$ay = a^{2j_0}b^{2k_0},\ by = a^{2m_0}b^{2n_0},\ cy = a^{2t_0}b^{2u_0}$$

for some $j_0, k_0, m_0, n_0, t_0, u_0 \in \mathbb{Z}_2$. Calculating $a(zy)$, $b(zy)$, and $c(zy)$, we get

$$\begin{aligned} a(zy) &= a^{2(jj_0+km_0)}b^{2(jk_0+kn_0)} = a^{2j_0}b^{2k_0} = ay, \\ b(zy) &= a^{2(mj_0+nm_0)}b^{2(mk_0+nn_0)}, \\ c(zy) &= a^{2(t_0+tj_0+um_0)}b^{2(u_0+tk_0+un_0)}. \end{aligned}$$

Then $zy = y$ if and only if $b(zy) = by$ and $c(zy) = cy$, i.e.

$$mj_0 \equiv mk_0 \equiv 0\,(\mathrm{mod}\,2),\; tj_0 + um_0 \equiv 0\,(\mathrm{mod}\,2),\; tk_0 + un_0 \equiv 0\,(\mathrm{mod}\,2). \tag{6.4}$$

It follows that $zy = y$ for each proper endomorphism y if and only if congruences (6.4) hold for each $j_0, k_0, m_0, n_0 \in \mathbb{Z}_2$. It is possible if and only if $m \equiv t \equiv u \equiv 0\,(\mathrm{mod}\,2)$, i.e. $m = 2m_1$, $t = 2t_1$, and $u = 2u_1$ for some $m_1, t_1, u_1 \in \mathbb{Z}_2$. Let us calculate z^2 in this case:

$$\begin{aligned} az^2 &= (a^j b^k)z = (a^j b^k)^j (a^m b^n)^k = a^{j^2+km} b^{k(j+n)} = a, \\ bz^2 &= (a^m b^n)z = (a^j b^k)^m (a^m b^n)^n = a^{m(j+n)} b^{km+n^2} = b, \\ cz^2 &= (ca^{2t_1} b^{2u_1})z = ca^{2t_1} b^{2u_1} \cdot (a^j b^k)^{2t_1} (a^m b^n)^{2u_1} \\ &= ca^{2t_1} b^{2u_1} \cdot a^{2t_1} b^{2u_1} = c, \end{aligned}$$

because

$$n^2 \equiv j^2 \equiv 1\,(\mathrm{mod}\,4),\; km \equiv m(j+n) \equiv k(j+n) \equiv 0\,(\mathrm{mod}\,4),$$
$$k \equiv m \equiv 0\,(\mathrm{mod}\,2),\; j \equiv n \equiv 1\,(\mathrm{mod}\,2).$$

Therefore, $z^2 = 1$ and property 4^0 is true. The necessity is proved.

Sufficiency. Let G be a finite group such that properties 1^0–4^0 hold. Our aim is to prove that $G \cong \mathcal{G}_{40}$. Property 3^0 implies that G does not split into non-trivial semidirect product $G = M \rtimes K$, because otherwise the projection x of G onto K is an idempotent of $\mathrm{End}(G)$ such that $x \neq 0$, $x \neq 1$.

We shall carry out further proof in a number of lemmas.

Lemma 6.1. *The group G is a non-Abelian 2-group. The group G has at least two elements of order 2.*

Proof. By property 1^0, $\widehat{g} = 1$ for each $2'$-element g of G. Hence all $2'$-elements of G belong into its centre $Z(G)$. Therefore, the group G splits into the direct product $G = G_{2'} \times G_2$ of its Hall $2'$-subgroup $G_{2'}$ and Sylow 2-subgroup G_2. Denote by z the projection of G onto its subgroup $G_{2'}$. By property 3^0, $z = 0$ or $z = 1$. Assume that $z = 1$. Then $G = G_{2'}$ is Abelian and, again by property 3^0, G is cyclic, i.e. $G = C_n$ for an odd integer n. By properties 1^0 and 2^0, we have $|\mathrm{End}(G)| = n = 256 + 64 = 320$. This contradicts the fact that n is odd. Hence $z = 0$ and G is a 2-group. The group G is non-Abelian, because otherwise, by property 3^0, G is cyclic and $|G| = |\mathrm{End}(G)| = 2^m = 320$ for an integer m, which is impossible.

To prove the last statement of the lemma, assume that G has only one element of order 2. By [15], Theorem 5.3.6, G is a generalized quaternion group because G is non-Abelian. This contradicts property 2^0 because a generalized quaternion group has only four proper endomorphisms ([7], Lemma 2). It follows that G has at least two elements of order 2. The lemma is proved.

The factor-group G/G' splits into a direct product

$$G/G' = \langle a_1 G' \rangle \times \ldots \times \langle a_n G' \rangle;\; a_1, \ldots, a_n \in G \setminus G'.$$

Denote by ε the canonical homomorphism $\varepsilon : G \longrightarrow G/G'$.

Lemma 6.2. $2 \leq n \leq 3$.

Proof. Note that G/G' is not cyclic ([15], Theorem 5.3.1). Hence $n \geq 2$. By Lemma 6.1, G has at least two elements of order two, for example b and c. We can assume that $c \in Z(G)$, i.e., $bc = cb$. Therefore, G has at least 4^n proper endomorphisms $z_{i_1 \ldots i_n j_1 \ldots j_n}$:

$$z_{i_1 \ldots i_n j_1 \ldots j_n} = \varepsilon \pi_{i_1 \ldots i_n j_1 \ldots j_n} : G \xrightarrow{\varepsilon} G/G' \xrightarrow{\pi_{i_1 \ldots i_n j_1 \ldots j_n}} \langle b, c \rangle, \tag{6.5}$$

$$(a_1 G')\pi_{i_1 \ldots i_n j_1 \ldots j_n} = b^{i_1} c^{j_1}, \ldots,\; (a_n G')\pi_{i_1 \ldots i_n j_1 \ldots j_n} = b^{i_n} c^{j_n}, \tag{6.6}$$

$i_1, \ldots, i_n,\, j_1, \ldots, j_n \in \mathbb{Z}_2$. By property 2^0, we have $4^n \leq 64 = 2^6$, i.e. $2 \leq n \leq 3$. The lemma is proved.

Lemma 6.3. *If* $x \in \mathrm{End}(G) \setminus \mathrm{Aut}(G)$, *then* $\mathrm{Im}x \subset G' \subset \mathrm{Ker}x$. *Each element of order two of G belongs into* G'.

Proof. Assume that $x \in \mathrm{End}(G) \setminus \mathrm{Aut}(G)$ and $z_{i_1 \ldots i_n j_1 \ldots j_n}$ are given by (6.5) and (6.6). By property 3^0,

$$\mathrm{Im}x \subset \cap_{y \in \mathrm{End}(G) \setminus \mathrm{Aut}(G)} \mathrm{Ker}y. \tag{6.7}$$

Since

$$\cap_{i_1 \ldots i_n j_1 \ldots j_n} \mathrm{Ker}\, z_{i_1 \ldots i_n j_1 \ldots j_n} = G',$$

(6.7) implies $\mathrm{Im}x \subset G' \subset \mathrm{Ker}x$. If $g \in G$ is an element of order two, then there exists a proper endomorphism y of G such that $\mathrm{Im}y = \langle g \rangle$. For example, $y = \varepsilon\pi$, where

$$\pi : G/G' \longrightarrow G,\ (a_1G')\pi = g,\ (a_iG')\pi = 1,\ i \neq 1.$$

By the first part of the proof, $\mathrm{Im}x = \langle g \rangle \subset G'$, i.e. $g \in G'$. The lemma is proved.

Lemma 6.4. *In the group the following properties hold*:

$$g^2 \in Z(G) \textit{ for each } g \in G,\ \ G' \subset Z(G).$$

Proof. Assume that $g \in G$. By Lemma 6.3,

$$h(\widehat{g} \cdot x) = (h \cdot [h, g])x = hx$$

for each $h \in G$ and $x \in \mathrm{End}(G) \setminus \mathrm{Aut}(G)$, i.e. $\widehat{g} \cdot x = x$ for each $x \in \mathrm{End}(G) \setminus \mathrm{Aut}(G)$. Property 4^0 implies that $\widehat{g}^2 = 1$, i.e. $g^2 \in Z(G)$. Therefore, all elements of the factor-group $G/Z(G)$ (except the unity element) are of order two. This implies that the factor-group $G/Z(G)$ is Abelian and $G' \subset Z(G)$. The lemma is proved.

Lemma 6.4 implies that $[g, h]^2 = 1$ for each $g, h \in G$, and, therefore,

$$g^2 = 1 \text{ for each } g \in G',$$

i.e. G' is an elementary Abelian group.

Lemma 6.5. $G' \cong C_2 \times C_2$ *and* $G/G' \cong C_2 \times C_2 \times C_2$.

Proof. Let us prove that $n = 3$. On the contrary, assume that $n = 2$, i.e.

$$G/G' = \langle a_1G' \rangle \times \langle a_2G' \rangle.$$

By Lemma 6.4, it is clear that $G' = \langle [a_1, a_2] \rangle$. Hence G' has only one element of order two. This contradicts Lemmas 6.1 and 6.3. Therefore $n = 3$ and

$$G/G' = \langle a_1G' \rangle \times \langle a_2G' \rangle \times \langle a_3G' \rangle.$$

By property 2^0, G has 64 proper endomorphisms. Since $n = 3$, all these proper endomorphisms are $z_{i_1 i_2 i_3 j_1 j_2 j_3}$ given by (6.5) and (6.6), where b and c are different elements of order two. By Lemma 6.3, $b \in G'$, $c \in G'$. If G' has an element d of order two such that $d \notin \langle b \rangle \times \langle c \rangle$, then there exists a proper endomorphism $y = \varepsilon\tau$ that does not have the form $z_{i_1 i_2 i_3 j_1 j_2 j_3}$:

$$G \xrightarrow{\varepsilon} G/G' \xrightarrow{\tau} \langle d \rangle,\ \ (a_1G')\tau = d,\ (a_2G')\tau = 1,\ (a_3G')\tau = 1.$$

Since G' is an elementary Abelian group, the obtained contradiction implies $G' \cong C_2 \times C_2$.

If there exists $i \in \{1, 2, 3\}$ such that $a_i^2 \notin G'$, then there exists a proper endomorphism $z = \varepsilon\mu$ that does not have the form $z_{i_1 i_2 i_3 j_1 j_2 j_3}$:

$$G \xrightarrow{\varepsilon} G/G' \xrightarrow{\mu} \langle d \rangle, \ (a_i G')\mu = a_i^{0(a_i)/4}, \ (a_j G')\mu = 1, \ i \neq j.$$

This contradicts the fact that all proper endomorphisms have the form $z_{i_1 i_2 i_3 j_1 j_2 j_3}$. Hence $a_i^2 \in G'$ for each i and $G/G' \cong C_2 \times C_2 \times C_2$. The lemma is proved.

It follows from Lemma 6.5 that $|G| = 32$ and $g^4 = 1$ for each $g \in G$. By Lemma 6.3, $g^2 \neq 1$ for each $g \in G \setminus G'$. Therefore, the group G has 3 elements of order two and 28 elements of order four. By [4], only the group $\mathscr{G}_{40}$ is a non-Abelian group of order 32, which has 2^8 automorphisms and has this order structure of its elements. Therefore, $G \cong \mathscr{G}_{40}$.

The sufficiency is proved. The theorem is proved.

Theorem 6.2. *The group $\mathscr{G}_{40}$ is determined by its endomorphism semigroup in the class of all groups.*

The proof of Theorem 6.2 is similar to that of Theorem 4.2.

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Maksimaalset alamrühma C_2^4 või C_4^2 omavate 32. järku rühmade endomorfismidest

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On tõestatud, et kõik 32. järku rühmad, mille üheks maksimaalseks alamrühmaks on $C_2 \times C_2 \times C_2 \times C_2$ või $C_4 \times C_4$, on määratud oma endomorfismipoolrühmadega kõigi rühmade klassis. Ühtlasi on antud mainitud rühmade kirjeldused nende endomorfismipoolrühmade kaudu.