Proceedings of the Estonian Academy of Sciences,
2014, **63**, 2, 121–132
doi: 10.3176/proc.2014.2.02
Available online at www.eap.ee/proceedings

MATHEMATICS

F-seminorms on generalized double sequence spaces defined by modulus functions

Enno Kolk[a] and Annemai Raidjõe[b*]

[a] Institute of Mathematics, University of Tartu, 50090 Tartu, Estonia; enno.kolk@ut.ee
[b] Institute of Mathematics, Tallinn University of Technology, 19086 Tallinn, Estonia

Received 15 November 2013, revised 28 November 2013, accepted 11 February 2014, available online 20 May 2014

Abstract. Using a double sequence of modulus functions and a solid double scalar sequence space, we determine F-seminorm and F-norm topologies for certain generalized linear spaces of double sequences. The main results are applied to the topologization of double sequence spaces related to 4-dimensional matrix methods of summability.

Key words: double sequence, F-seminorm, F-norm, matrix method, modulus function, paranorm, sequence space.

1. INTRODUCTION

Let $\mathbb{N} = \{1, 2, \dots\}$ and let $\mathbb{K}$ be the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. We specify the domains of indices only if they are different from $\mathbb{N}$. By the symbol ι we denote the identity mapping $\iota(z) = z$. We also use the notation $\mathbb{R}^+ = [0, \infty)$.

Let $\mathbf{e}^n = (e^n_k)_{k\in\mathbb{N}}$ $(n \in \mathbb{N})$ be the sequences, where $e^n_k = 1$ if $k = n$ and $e^n_k = 0$ otherwise. We also consider the corresponding double sequences $\mathbf{e}^{n(2)} = (e^n_{ki})$ $(n \in \mathbb{N})$ such that, for all $i \in \mathbb{N}$, $e^n_{ki} = 1$ if $k = n$ and $e^n_{ki} = 0$ if $k \neq n$.

In all definitions which contain infinite series we tacitly assume the convergence of these series.

An F-space is usually understood as a complete metrizable topological vector space over $\mathbb{K}$. It is known that the topology of an F-space E can be given by an F-*norm*, i.e., by a functional $g : E \to \mathbb{R}^+$ with the axioms (see [6, p. 13])
(N1) $g(0) = 0$,
(N2) $g(x+y) \le g(x) + g(y) \quad (x, y \in E)$,
(N3) $|\alpha| \le 1 \;\; (\alpha \in \mathbb{K}),\; x \in E \implies g(\alpha x) \le g(x)$,
(N4) $\lim_n \alpha_n = 0 \;\; (\alpha_n \in \mathbb{K}),\; x \in E \implies \lim_n g(\alpha_n x) = 0$,
(N5) $g(x) = 0 \implies x = 0$.

A functional g with the axioms (N1)–(N4) is called an F-*seminorm*. A *paranorm* on E is defined as a functional $g : E \to \mathbb{R}^+$ satisfying the axioms (N1), (N2), and
(N6) $g(-x) = g(x) \quad (x \in E)$,
(N7) $\lim_n \alpha_n = \alpha \quad (\alpha_n, \alpha \in \mathbb{K}),\; \lim_n g(x_n - x) = 0 \quad (x_n, x \in E) \implies \lim_n g(\alpha_n x_n - \alpha x) = 0$.

A *seminorm* on E is a functional $g : E \to \mathbb{R}$ with the axioms (N1), (N2), and
(N8) $g(\alpha x) = |\alpha| g(x) \quad (\alpha \in \mathbb{K},\; x \in E)$.

* Corresponding author, annemai.raidjoe@ttu.ee

An F-seminorm (paranorm, seminorm) g is called *total* if (N5) holds. So, an F-norm (norm) is a total F-seminorm (seminorm).

Unlike the module $|\cdot|$, following also [8], the seminorm of an element $x \in E$ is often denoted by $\dot{|}x\dot{|}$.

It is known (see [8, Remark 1]) that F-seminorms coincide with paranorms satisfying (N3).

Let $\mathbf{X}^2$ be a double sequence of seminormed linear spaces $\left(X_{ki}, \dot{|}\cdot\dot{|}_{ki}\right)$ $(k,i \in \mathbb{N})$. Then the set $s^2(\mathbf{X}^2)$ of all double sequences $\mathbf{x}^2 = (x_{ki})$, $x_{ki} \in X_{ki}$ $(k,i \in \mathbb{N})$, together with coordinatewise addition and scalar multiplication, is a linear space (over $\mathbb{K}$). Any linear subspace of $s^2(\mathbf{X}^2)$ is called a *generalized double sequence space* (GDS space). If $\left(X_{ki}, \dot{|}\cdot\dot{|}_{ki}\right) = \left(X, \dot{|}\cdot\dot{|}\right)$ $(k,i \in \mathbb{N})$, then we write X^2 instead of $\mathbf{X}^2$. In the case $X = \mathbb{K}$ we omit the symbol X^2 in notation. So, for example, s^2 denotes the linear space of all $\mathbb{K}$-valued double sequences $\mathbf{u}^2 = (u_{ki})$. By s we denote the linear space of all single $\mathbb{K}$-valued sequences $\mathbf{u} = (u_k)$. As usual, linear subspaces of s^2 are called *double sequence spaces* (DS spaces) and linear subspaces of s are called *sequence spaces*. Well-known sequence spaces are the sets ℓ_∞, c, c_0, and ℓ_p $(p > 0)$ of all bounded, convergent, convergent to zero, and absolutely p-summable number sequences, respectively. Examples of DS spaces are

$$\tilde{s}^2 = \{\mathbf{u}^2 \in s^2 : \tilde{u}_k = \sup_i |u_{ki}| < \infty \quad (k \in \mathbb{N})\}$$

and

$$\tilde{\lambda}^2 = \{\mathbf{u}^2 \in \tilde{s}^2 : \tilde{\mathbf{u}} = (\tilde{u}_k) \in \lambda\}$$

with $\lambda \in \{\ell_\infty, c_0, \ell_p\}$. Double sequence spaces are also the sets c^2 and c_0^2 of all double scalar sequences which, respectively, converge and converge to zero in the Pringsheim sense. Recall that a sequence (u_{ki}) is said to be *Pringsheim convergent* to a number L if for every $\varepsilon > 0$ there exists an index n_0 such that $|u_{ki} - L| < \varepsilon$ whenever $k,i > n_0$ (see [12] or [18, Chapter 8]). In this case we write $\text{P-lim}_{k,i} u_{ki} = L$.

The idea of a modulus function was structured by Nakano [11]. Following Ruckle [14] and Maddox [9], we say that a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a *modulus function* (or, simply, a *modulus*), if

(M1) $\phi(t) = 0 \iff t = 0$,

(M2) $\phi(t+u) \le \phi(t) + \phi(u)$,

(M3) ϕ is non-decreasing,

(M4) ϕ is continuous from the right at 0.

For example, the function $\iota^p(t) = t^p$ is an unbounded modulus for $p \le 1$ and the function $\phi(t) = t/(1+t)$ is a bounded modulus.

Since $|\phi(t) - \phi(u)| \le \phi(|t-u|)$ by (M1)–(M3), the moduli are continuous everywhere on $\mathbb{R}^+$. We also remark that the modulus functions are the same as the moduli of continuity (see [5, p. 866]).

A GDS space $\Lambda(\mathbf{X}^2) \subset s^2(\mathbf{X}^2)$ is called *solid* if $(y_{ki}) \in \Lambda(\mathbf{X}^2)$ whenever $(x_{ki}) \in \Lambda(\mathbf{X}^2)$ and $\dot{|}y_{ki}\dot{|}_{ki} \le \dot{|}x_{ki}\dot{|}_{ki}$ $(k,i \in \mathbb{N})$. For example, it is not difficult to see that the sets

$$\tilde{s}^2\left(\mathbf{X}^2\right) = \left\{\mathbf{x}^2 \in s^2(\mathbf{X}^2) : \sup_i \dot{|}x_{ki}\dot{|}_{ki} < \infty \ \ (k \in \mathbb{N})\right\},$$

$$\Lambda\left(\mathbf{\Phi}, \mathbf{X}^2\right) = \left\{\mathbf{x}^2 \in s^2(\mathbf{X}^2) : \mathbf{\Phi}(\mathbf{x}^2) = \left(\phi_{ki}\left(\dot{|}x_{ki}\dot{|}_{ki}\right)\right) \in \Lambda\right\},$$

and $\Lambda(\mathbf{\Phi}, \mathbf{X}^2) \cap \tilde{s}^2(\mathbf{X}^2)$ are solid GDS spaces if $\Lambda \subset s^2$ is a solid DS space and $\mathbf{\Phi} = (\phi_{ki})$ is a double sequence of moduli.

Our aim is to determine F-seminorm topologies for GDS spaces of sequences $\mathbf{x}^2 \in s_T^2(\mathbf{X}^2)$ with $T\mathbf{x}^2$ in $\Lambda(\mathbf{\Phi}, \mathbf{Y}^2)$ or in $\Lambda(\mathbf{\Phi}, \mathbf{Y}^2) \cap \tilde{s}^2(\mathbf{Y}^2)$, where $\mathbf{Y}^2$ is another double sequence of seminormed linear spaces, $T : s_T^2(\mathbf{X}^2) \to s^2(\mathbf{Y}^2)$ is a linear operator defined on a linear subspace $s_T^2(\mathbf{X}^2)$ of $s^2(\mathbf{X}^2)$ and the solid DS space Λ is topologized by an absolutely monotone F-seminorm. Similar theorems have been proved earlier in [7,10,13,16] for analogical sets of single number sequences in the case $T = \iota$. The results of this paper are applied to the topologization of GDS spaces related to 4-dimensional matrix methods of summability. Some special cases of such spaces are considered, for example, in [1,3,4,15,17].

2. MAIN THEOREMS

Let $\lambda \subset s$ be a sequence space, $\Lambda \subset s^2$ be a DS space, and let $\mathbf{e}^k$, $\mathbf{e}^{k(2)}$ $(k \in \mathbb{N})$ be sequences defined above. Recall that an F-seminormed space (λ, g) is called an AK-*space*, if λ contains the sequences $\mathbf{e}^k$ $(k \in \mathbb{N})$ and for any $\mathbf{u} = (u_k) \in \lambda$ we have $\lim_n \mathbf{u}^{[n]} = \mathbf{u}$, where $\mathbf{u}^{[n]} = \sum_{k=1}^n u_k \mathbf{e}^k$. Generalizing this definition, we say that an F-seminormed DS space (Λ, g) is an AK-*space* if Λ contains the sequences $\mathbf{e}^{k(2)}$ $(k \in \mathbb{N})$ and for any $\mathbf{u}^2 = (u_{ki}) \in \Lambda$ we have $\lim_n \mathbf{u}^{2[n]} = \mathbf{u}^2$, where $\mathbf{u}^{2[n]} = \sum_{k=1}^n \mathbf{u}_k \mathbf{e}^{k(2)}$ with $\mathbf{u}_k = (u_{ki})_{i \in \mathbb{N}}$ and $\mathbf{u}_k \mathbf{e}^{k(2)} = (u_{ki} e^k_{ji})_{j,i \in \mathbb{N}}$. It is not difficult to see that $\tilde{c}_0^2$ is the AK-space with respect to norm $\|\mathbf{u}^2\|_\infty = \sup_{ki} |u_{ki}|$.

An F-seminorm g on a sequence space $\lambda \subset s$ is said to be *absolutely monotone* if for all $\mathbf{u} = (u_k)$ and $\mathbf{v} = (v_k)$ from λ with $|v_k| \leq |u_k|$ $(k \in \mathbb{N})$, we have $g(\mathbf{v}) \leq g(\mathbf{u})$. Analogously, an F-seminorm g on a GDS space $\Lambda(\mathbf{X}^2) \subset s^2(\mathbf{X}^2)$ is said to be *absolutely monotone* if for all $\mathbf{x}^2 = (x_{ki})$ and $\mathbf{y}^2 = (y_{ki})$ from $\Lambda(\mathbf{X}^2)$ with $|y_{ki}|_{ki} \leq |x_{ki}|_{ki}$ $(k, i \in \mathbb{N})$ we have $g(\mathbf{y}^2) \leq g(\mathbf{x}^2)$.

Soomer [16] and Kolk [7] proved that if a solid sequence space $\lambda \subset s$ is topologized by an absolutely monotone F-seminorm (or paranorm) g and $\Phi = (\phi_k)$ is a sequence of moduli, then the solid sequence space

$$\lambda(\Phi) = \{\mathbf{u} = (u_k) \in s : \Phi(\mathbf{u}) = (\phi_k(|u_k|)) \in \lambda\}$$

may be topologized by the absolutely monotone F-seminorm (paranorm)

$$g_\Phi(\mathbf{u}) = g(\Phi(\mathbf{u})) \ \ (\mathbf{u} \in \lambda(\Phi))$$

whenever either (λ, g) is an AK-space or the sequence Φ satisfies one of the two equivalent conditions

(M5) there exist a function ν and a number $\delta > 0$ such that $\lim_{u \to 0+} \nu(u) = 0$ and $\phi_k(ut) \leq \nu(u)\phi_k(t)$ $(0 \leq u < \delta,\ t \geq 0,\ k \in \mathbb{N})$,

(M6) $\lim\limits_{u \to 0+} \sup\limits_{t>0} \sup\limits_{k} \dfrac{\phi_k(ut)}{\phi_k(t)} = 0.$

This result was generalized in [10] and [13] to the sequence space

$$\Lambda(\mathbf{\Phi}) = \{\mathbf{u} \in s : \mathbf{\Phi}(\mathbf{u}) = (\phi_{ki}(|u_k|)) \in \Lambda\}$$

defined by means of a solid DS space Λ and a double sequence of moduli $\mathbf{\Phi} = (\phi_{ki})$. Thereby, in the case of AK-space (Λ, g) it is assumed that $\mathbf{\Phi}$ satisfies the conditions

(M7) $\tilde{\phi}_k(t) = \sup\limits_i \phi_{ki}(t) < \infty$ $(t \in \mathbb{R}^+,\ k \in \mathbb{N})$,

(M8) $\lim\limits_{t \to 0+} \tilde{\phi}_k(t) = 0$ $(k \in \mathbb{N})$.

In the following we extend these results to the generalized double sequence spaces defined by means of a linear operator $T : s^2_T(\mathbf{X}^2) \to s^2(\mathbf{Y}^2)$ with $T\mathbf{x}^2 = (T_{ki}\mathbf{x}^2)$, and by means of the set $\tilde{s}^2(\mathbf{Y}^2)$, where $\mathbf{Y}^2$ is another double sequence of seminormed spaces $(Y_{ki}, |\cdot|_{ki})$ $(k, i \in \mathbb{N})$.

Theorem 1. *Let $\Lambda \subset s^2$ be a solid DS space which is topologized by an absolutely monotone F-seminorm g. If the double sequence of moduli $\mathbf{\Phi} = (\phi_{ki})$ satisfies the condition*

(M5') *there exist a function ν and a number $\delta > 0$ such that $\lim_{u \to 0+} \nu(u) = 0$ and $\phi_{ki}(ut) \leq \nu(u)\phi_{ki}(t)$ $(k, i \in \mathbb{N},\ 0 < u < \delta,\ t > 0)$,*

then the GDS space

$$\Lambda\left(\mathbf{\Phi}, T, \mathbf{X}^2, \mathbf{Y}^2\right) = \left\{\mathbf{x}^2 \in s^2_T(\mathbf{X}^2) : T\mathbf{x}^2 \in \Lambda(\mathbf{\Phi}, \mathbf{Y}^2)\right\}$$

may be topologized by the F-seminorm

$$g_{\mathbf{\Phi},T}\left(\mathbf{x}^2\right) = g\left(\mathbf{\Phi}\left(T\mathbf{x}^2\right)\right).$$

Thereby, if g is an F-norm in Λ, the spaces Y_{ki} are normed and T satisfies the condition

$$T\mathbf{x}^2 = 0 \implies \mathbf{x}^2 = 0, \tag{1}$$

then $g_{\mathbf{\Phi},T}$ is an F-norm in $\Lambda\left(\mathbf{\Phi}, T, \mathbf{X}^2, \mathbf{Y}^2\right)$.

The F-seminorm $g_{\mathbf{\Phi},T}$ is absolutely monotone if

$$\dot{|}y_{ki}\dot{|}_{ki} \le \dot{|}x_{ki}\dot{|}_{ki} \quad (k,i \in \mathbb{N}) \implies \dot{|}T_{ki}\mathbf{y}^2\dot{|}_{ki} \le \dot{|}T_{ki}\mathbf{x}^2\dot{|}_{ki} \quad (k,i \in \mathbb{N}). \tag{2}$$

Proof. Similarly to the proof of Theorem 2.2 [10], using also the linearity of T, it is not difficult to show that the functional $g_{\mathbf{\Phi},T}$ satisfies the axioms (N1)–(N3). To prove (N4), let $\lim_n \alpha_n = 0$. Then there exists an index n_0 with $|\alpha_n| < \delta$ for $n \ge n_0$. Since by (M5') we have

$$\phi_{ki}\left(\dot{|}T_{ki}(\alpha_n\mathbf{x}^2)\dot{|}_{ki}\right) \le \nu(|\alpha_n|)\phi_{ki}\left(\dot{|}T_{ki}\mathbf{x}^2\dot{|}_{ki}\right) \quad (k,i \in \mathbb{N})$$

and g is absolutely monotone,

$$g\left(\mathbf{\Phi}\left(T\left(\alpha_n\mathbf{x}^2\right)\right)\right) \le g\left(\nu(|\alpha_n|)\mathbf{\Phi}\left(T\mathbf{x}^2\right)\right) \quad (n \ge n_0).$$

But this yields $\lim_n g_{\mathbf{\Phi},T}(\alpha_n\mathbf{x}^2) = 0$ by $\lim_n \nu(|\alpha_n|) = 0$. Thus (N4) holds and $g_{\mathbf{\Phi},T}$ is an F-seminorm on the GDS space $\Lambda(\mathbf{\Phi}, T, \mathbf{X}^2, \mathbf{Y}^2)$.

Now, let g be an F-norm on Λ and let the spaces Y_{ki} be normed by the norms $\|\cdot\|_{ki}$. If $g_{\mathbf{\Phi},T}(\mathbf{x}^2) = 0$, then, using also (M1), we have

$$\|T_{ki}\mathbf{x}^2\|_{ki} = 0 \ \ (k,i \in \mathbb{N})$$

which gives $\mathbf{x}^2 = 0$ by (1). So, $g_{\mathbf{\Phi},T}$ is an F-norm in this case.

Finally, let T satisfy (2). If $\dot{|}y_{ki}\dot{|}_{ki} \le \dot{|}x_{ki}\dot{|}_{ki}$ $(k,i \in \mathbb{N})$, then

$$\phi_{ki}\left(\dot{|}T_{ki}\mathbf{y}^2\dot{|}_{ki}\right) \le \phi_{ki}\left(\dot{|}T_{ki}\mathbf{x}^2\dot{|}_{ki}\right) \quad (k,i \in \mathbb{N})$$

and since g is absolutely monotone,

$$g_{\mathbf{\Phi},T}(\mathbf{y}^2) = g(\mathbf{\Phi}(T\mathbf{y}^2)) \le g(\mathbf{\Phi}(T\mathbf{x}^2)) = g_{\mathbf{\Phi},T}(\mathbf{x}^2).$$

Consequently, F-seminorm (F-norm) $g_{\mathbf{\Phi},T}$ is absolutely monotone if (2) holds. □

Remark 1. It is easy to see that the condition (M5') in Theorem 1 may be replaced by the equivalent condition

(M6') $\displaystyle\lim_{u\to 0+} \sup_{t>0} \sup_{k,i} \frac{\phi_{ki}(ut)}{\phi_{ki}(t)} = 0.$

Theorem 2. *Let $\Lambda \subset s^2$ be a solid AK-space with respect to an absolutely monotone F-seminorm g. If the double sequence of moduli $\mathbf{\Phi} = (\phi_{ki})$ satisfies* (M7) *and* (M8), *then the GDS space*

$$\Lambda\left(\mathbf{\Phi}, \tilde{T}, \mathbf{X}^2, \mathbf{Y}^2\right) = \left\{\mathbf{x}^2 \in s_T^2(\mathbf{X}^2) : T\mathbf{x}^2 \in \Lambda(\mathbf{\Phi}, \mathbf{Y}^2) \cap \tilde{s}^2(\mathbf{Y}^2)\right\}$$

may be topologized by the F-seminorm $g_{\mathbf{\Phi},T}$. Thereby, if g is an F-norm in Λ, the spaces Y_{ki} are normed and T satisfies (1), *then $g_{\mathbf{\Phi},T}$ is an F-norm on $\Lambda(\mathbf{\Phi}, \tilde{T}, \mathbf{X}^2, \mathbf{Y}^2)$.*

The F-seminorm $g_{\mathbf{\Phi},T}$ is absolutely monotone in $\Lambda(\mathbf{\Phi}, \tilde{T}, \mathbf{X}^2, \mathbf{Y}^2)$ whenever T satisfies (2).

Proof. The functional $g_{\mathbf{\Phi},T} : \Lambda(\mathbf{\Phi}, \tilde{T}, \mathbf{X}^2, \mathbf{Y}^2) \to \mathbb{K}$ obviously satisfies the axioms (N1)–(N3). To prove (N4), let $\lim_n \alpha_n = 0$ and let $\mathbf{x}^2$ be an arbitrary element from the space $\Lambda(\mathbf{\Phi}, \tilde{T}, \mathbf{X}^2, \mathbf{Y}^2)$. Then $\mathbf{\Phi}(T\mathbf{x}^2) \in \Lambda$ and since Λ is an AK-space,

$$\lim_n g\left(\mathbf{\Phi}(T\mathbf{x}^2) - \mathbf{\Phi}(T\mathbf{x}^2)^{[n]}\right) = 0. \tag{3}$$

Using the equality

$$\mathbf{\Phi}(T\mathbf{x}^2) - \mathbf{\Phi}(T\mathbf{x}^2)^{[n]} = \mathbf{\Phi}\left(T\mathbf{x}^2 - (T\mathbf{x}^2)^{[n]}\right),$$

by (3) we can find, for fixed $\varepsilon > 0$, an index m such that

$$g\left(\mathbf{\Phi}\left(T\mathbf{x}^2 - (T\mathbf{x}^2)^{[m]}\right)\right) < \varepsilon/2. \tag{4}$$

The double sequence $T\mathbf{x}^2 \in \tilde{s}^2(\mathbf{Y}^2)$ determines the single sequence $(\bar{z}_k)$ of numbers $\bar{z}_k = \sup_i \left|T_{ki}\mathbf{x}^2\right|_{ki}$ $(k \in \mathbb{N})$. Since

$$\lim_n \tilde{\phi}_k(|\alpha_n \bar{z}_k|) = 0 \quad (k \in \mathbb{N})$$

by (M8), and g satisfies (N4), we have that

$$\lim_n g\left(\tilde{\phi}_k\left(|\alpha_n \bar{z}_k|\right)\mathbf{e}^{k(2)}\right) = 0 \quad (k \in \mathbb{N}). \tag{5}$$

Further, since g satisfies (N2) and it is absolutely monotone, we may write

$$\begin{aligned} g\left(\mathbf{\Phi}\left(T\left(\alpha_n\mathbf{x}^2\right)\right)^{[m]}\right) &= g\left(\sum_{k=1}^{m}\left(\phi_{ki}\left(\left|\alpha_n T_{ki}\mathbf{x}^2\right|_{ki}\right)\right)_i \mathbf{e}^{k(2)}\right) \\ &\le \sum_{k=1}^{m} g\left(\left(\phi_{ki}\left(\left|\alpha_n T_{ki}\mathbf{x}^2\right|_{ki}\right)\right)_i \mathbf{e}^{k(2)}\right) \\ &\le \sum_{k=1}^{m} g\left(\tilde{\phi}_k\left(|\alpha_n \bar{z}_k|\right)\mathbf{e}^{k(2)}\right). \end{aligned}$$

This yields

$$\lim_n g\left(\mathbf{\Phi}\left(T\left(\alpha_n\mathbf{x}^2\right)\right)^{[m]}\right) = 0$$

because of (5). Thus there exists an index n_0 such that, for all $n \ge n_0$,

$$|\alpha_n| \le 1 \;\text{ and }\; g\left(\mathbf{\Phi}\left(|\alpha_n|\left(T\mathbf{x}^2\right)^{[m]}\right)\right) < \varepsilon/2. \tag{6}$$

Now, by (4) and (6) we get

$$\begin{aligned} g_{\mathbf{\Phi},T}(\alpha_n\mathbf{x}^2) &= g(\mathbf{\Phi}(T(\alpha_n\mathbf{x}^2))) \\ &\le g\left(\mathbf{\Phi}\left(|\alpha_n|(T\mathbf{x}^2 - (T\mathbf{x}^2)^{[m]})\right)\right) + g\left(\mathbf{\Phi}\left(|\alpha_n|(T\mathbf{x}^2)^{[m]}\right)\right) \\ &\le g\left(\mathbf{\Phi}\left(T\mathbf{x}^2 - (T\mathbf{x}^2)^{[m]}\right)\right) + g\left(\mathbf{\Phi}\left(|\alpha_n|(T\mathbf{x}^2)^{[m]}\right)\right) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for $n \ge n_0$. Hence $\lim_n g_{\mathbf{\Phi},T}(\alpha_n\mathbf{x}^2) = 0$, i.e., (N4) is true for $g_{\mathbf{\Phi},T}$.

Similarly to the proof of Theorem 1 we can see that the F-seminorm $g_{\mathbf{\Phi},T}$ is absolutely monotone whenever (2) holds, and $g_{\mathbf{\Phi},T}$ is an F-norm if g is an F-norm, the spaces X_{ki} are normed and (1) is true. □

Remark 2. The investigations of Basu and Srivastava [1] contain, for one modulus ϕ and for a sequence $\mathbf{p}^2 = (p_{ki})$ of positive numbers $p_{ki} \le 1$, the GDS space $\Lambda(\mathbf{\Phi}, X^2)$, where $\phi_{ki}(t) = [\phi(t)]^{p_{ki}}$. They assert (see [1, Theorem 3.2]) that if Λ is topologized by an absolutely monotone paranorm g, then

$$g_{\mathbf{\Phi}}(\mathbf{x}^2) = g(\mathbf{\Phi}(\mathbf{x}^2))$$

is a paranorm on $\Lambda(\boldsymbol{\Phi}, X^2)$ whenever $\inf_{k,i} p_{ki} > 0$. But this is not true in general. Indeed, if ϕ is a bounded modulus, $p_{ki} = 1$, and the solid sequence space $\tilde{\ell}_\infty^2$ is topologized by the absolutely monotone norm $g(\mathbf{u}^2) = \sup_{k,i} |u_{ki}|$, then the set

$$\tilde{\ell}_\infty^2(\phi, X^2) = \left\{ \mathbf{x}^2 \in s^2(X^2) : \sup_{k,i} \phi\left(\left|x_{ki}\right|\right) < \infty \right\}$$

coincides with $s^2(X^2)$. Consequently, $\tilde{\ell}_\infty^2(\phi, X^2)$ contains an unbounded sequence $\mathbf{z}^2 = (z_{ki})$ with $z_{ki} \neq 0$ such that for a subsequence of indices (k_j) the equality $\lim_j \left|z_{k_j,k_j}\right| = \infty$ holds. Then, defining

$$\alpha_n = \begin{cases} \left(\left|z_{k_j,k_j}\right|\right)^{-1}, & \text{if } n = k_j \quad (j \in \mathbb{N}) \\ 0 & \text{otherwise,} \end{cases}$$

we get the sequence (α_n) with $\lim_n \alpha_n = 0$. Since

$$\phi\left(\left|\alpha_{k_j} z_{k_j,k_j}\right|\right) = \phi(1) > 0 \quad (j \in \mathbb{N}),$$

we have that

$$\lim_n g_{\boldsymbol{\Phi}}(\alpha_n \mathbf{z}^2) = \lim_n \sup_{k,i} \phi\left(\left|\alpha_n z_{k,i}\right|\right) \neq 0.$$

Thus $g_{\boldsymbol{\Phi}}$ does not satisfy the axiom (N4) and, consequently, it is not a paranorm on the GDS space $\tilde{\ell}_\infty^2(\phi, X^2)$ if the modulus ϕ is bounded. Theorem 1 (for $T = \iota$) shows that if the solid double sequence space Λ is topologized by an absolutely monotone F-seminorm (or a paranorm with (N3)) g, then

$$g_\phi\left(\mathbf{x}^2\right) = g\left(\left(\phi\left(\left|x_{ki}\right|_{ki}\right)\right)_{k,i \in \mathbb{N}}\right)$$

is an absolutely monotone F-seminorm (paranorm) on the GDS space

$$\Lambda(\phi, \mathbf{X}^2) = \left\{ \mathbf{x}^2 \in s^2(\mathbf{X}^2) : \left(\phi\left(\left|x_{ki}\right|_{ki}\right)\right)_{k,i \in \mathbb{N}} \in \Lambda \right\}$$

whenever the modulus ϕ satisfies the condition

(M5°) there exist a function ν and a number $\delta > 0$ such that $\lim_{u \to 0+} \nu(u) = 0$ and $\phi(ut) \leq \nu(u)\phi(t)$ $(0 \leq u < \delta,\ t \geq 0)$,

or the condition (see Remark 1)

(M6°) $\lim_{u \to 0+} \sup_{t>0} \dfrac{\phi(ut)}{\phi(t)} = 0.$

These conditions clearly fail if ϕ is bounded, since by $\sup_{t>0} \phi(t) = M < \infty$ we have

$$\sup_{t>0} \frac{\phi(ut)}{\phi(t)} \geq M^{-1} \sup_{t>0} \phi(ut) = 1$$

for any fixed $u > 0$.

It should be noted that the same remark is true concerning [2, Theorem 3.1].

3. SOME APPLICATIONS

Let $\mathscr{A} = (a_{mnki})$ be a non-negative 4-dimensional matrix, i.e., $a_{mnki} \geq 0 \quad (m,n,k,i \in \mathbb{N})$. By $\mathscr{I}$ we denote the 4-dimensional unit matrix. We say that $\mathscr{A}$ is *essentially positive* if for any $k,i \in \mathbb{N}$ there exist indices m_k and n_i such that $a_{m_k,n_i,k,i} > 0$. A sequence $\mathbf{u}^2 = (u_{ki}) \in s^2$ is called *strongly $\mathscr{A}$-summable with index $p \geq 1$ to a number L* if P-$\lim_{m,n} \sum_{k,i} a_{mnki}|u_{ki} - L|^p = 0$, and *strongly $\mathscr{A}$-bounded with index p* if $\sup_{m,n} \sum_{k,i} a_{mnki}|u_{ki}|^p < \infty$. It is clear that the set $c_0^2[\mathscr{A}]^p$ of all strongly $\mathscr{A}$-summable with index p to zero sequences and the set $\tilde{\ell}_\infty^2[\mathscr{A}]^p$ of all strongly $\mathscr{A}$-bounded with index p sequences are solid linear spaces. Since the Pringsheim convergent double sequences are not necessarily bounded, $c_0^2[\mathscr{A}]^p$ is not a subset of $\tilde{\ell}_\infty^2[\mathscr{A}]^p$ in general. A subset of $\tilde{\ell}_\infty^2[\mathscr{A}]^p$ represents the DS space

$$\tilde{c}_0^2[\mathscr{A}]^p = \left\{ \mathbf{u}^2 \in s^2 : \lim_m \sup_n \sum_{k,i} a_{mnki}|u_{ki}|^p = 0 \right\}.$$

Denoting $bc_0^2[\mathscr{A}]^p = c_0^2[\mathscr{A}]^p \cap \tilde{\ell}_\infty^2[\mathscr{A}]^p$, we also have $\tilde{c}_0^2[\mathscr{A}]^p \subset bc_0^2[\mathscr{A}]^p$.

It is not difficult to see that the functional

$$g_{\mathscr{A}}^p(\mathbf{u}^2) = \sup_{m,n} \left(\sum_{k,i} a_{mnki}|u_{ki}|^p \right)^{1/p}$$

is a seminorm on $\tilde{\ell}_\infty^2[\mathscr{A}]^p$, it is a norm if $\mathscr{A}$ is essentially positive.

A natural generalization of DS spaces $\tilde{\ell}_\infty^2[\mathscr{A}]^p$, $\tilde{c}_0^2[\mathscr{A}]^p$, and $bc_0^2[\mathscr{A}]^p$ is related to an arbitrary solid F-seminormed (or seminormed) sequence space (Λ, g_Λ). It is easy to see that the set

$$\Lambda[\mathscr{A}]^p = \left\{ \mathbf{u}^2 \in s^2 : \mathscr{A}^{1/p}\left(|\mathbf{u}^2|^p\right) = \left(\left(\sum_{k,i} a_{mnki}|u_{ki}|^p \right)^{1/p} \right)_{m,n\in\mathbb{N}} \in \Lambda \right\}$$

is a solid linear subspace of s^2. In addition, if the F-seminorm (seminorm) g_Λ is absolutely monotone, then the functional

$$g_{\Lambda,\mathscr{A}}^p(\mathbf{u}^2) = g_\Lambda\left(\mathscr{A}^{1/p}\left(|\mathbf{u}^2|^p\right) \right)$$

defines an F-seminorm (seminorm) on $\Lambda[\mathscr{A}]^p$. At that, if $\mathscr{A}$ is essentially positive, then $g_{\Lambda,\mathscr{A}}^p$ is an F-norm (a norm) whenever the space Λ is F-normed (normed).

Let ϕ be a modulus function and let $\mathbf{p}^2 = (p_{ki}) \in \tilde{\ell}_\infty^2$ with $r = \max\{1, \sup_{k,i} p_{ki}\}$. Some sets of sequences $\mathbf{x}^2 = (x_{ki}) \in s^2(X^2)$, such that the sequence $\left(\left(\phi\left(|x_{ki}| \right) \right)^{p_{ki}} \right)$ belongs to $\tilde{\ell}_\infty^2[\mathscr{A}]^1$, $\tilde{c}_0^2[\mathscr{A}]^1$, or $bc_0^2[\mathscr{A}]^1$, are studied in [1,3,4,15]. These investigations lead us to the following, more general, notion of GDS spaces. For an arbitrary 4-dimensional matrix $\mathscr{B} = (b_{kilj})$ let $s_{\mathscr{B}}^2(X^2)$ be the set of all sequences $\mathbf{x}^2 = (x_{ki}) \in s^2(X^2)$ such that the series $\mathscr{B}_{ki}\mathbf{x}^2 = \sum_{lj} b_{kilj}x_{lj}$ converge. Let $\mathscr{B}\mathbf{x}^2 = (\mathscr{B}_{ki}\mathbf{x}^2)$. Using also a double sequence of moduli $\mathbf{\Phi} = (\phi_{ki})$ and a solid DS space Λ, we consider the sets

$$\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \mathscr{B}, X^2] = \left\{ \mathbf{x}^2 \in s_{\mathscr{B}}^2(X^2) : \mathscr{A}^{1/r}\left(\mathbf{\Phi}^{\mathbf{p}^2}\left(\mathscr{B}\mathbf{x}^2 \right) \right) \in \Lambda \right\},$$

$$\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \tilde{\mathscr{B}}, X^2] = \left\{ \mathbf{x}^2 \in s_{\mathscr{B}}^2(X^2) : \mathscr{B}\mathbf{x}^2 \in \tilde{s}^2(X^2) \text{ and } \mathscr{A}^{1/r}\left(\mathbf{\Phi}^{\mathbf{p}^2}\left(\mathscr{B}\mathbf{x}^2 \right) \right) \in \Lambda \right\},$$

where

$$\mathscr{A}^{1/r}\left(\mathbf{\Phi}^{\mathbf{p}^2}\left(\mathscr{B}\mathbf{x}^2 \right) \right) = \left(\left(\sum_{k,i} a_{mnki} \left(\phi_{ki}\left(\left| \sum_{l,j} b_{kilj}x_{lj} \right| \right) \right)^{p_{ki}} \right)^{1/r} \right)_{m,n\in\mathbb{N}}.$$

The following representations of these sets are useful. Using the equality $p_{ki} = (p_{ki}/r)r$ and denoting by $\mathbf{\Phi}^{\mathbf{p}^2/r}$ the sequence of moduli $\phi_{ki}^{\mathbf{p}^2/r}(t) = (\phi_{ki}(t))^{p_{ki}/r}$, we may write

$$\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \mathscr{B}, X^2] = \left\{ \mathbf{x}^2 \in s^2_{\mathscr{B}}(X^2) : \mathbf{\Phi}^{\mathbf{p}^2/r}\left(\mathscr{B}\mathbf{x}^2\right) \in \Lambda[\mathscr{A}]^r \right\}, \tag{7}$$

$$\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \tilde{\mathscr{B}}, X^2] = \left\{ \mathbf{x}^2 \in s^2_{\mathscr{B}}(X^2) : \mathscr{B}\mathbf{x}^2 \in \tilde{s}^2(X^2) \text{ and } \mathbf{\Phi}^{\mathbf{p}^2/r}\left(\mathscr{B}\mathbf{x}^2\right) \in \Lambda[\mathscr{A}]^r \right\}. \tag{8}$$

Since the DS space $\Lambda[\mathscr{A}]^r$ is solid and the summability operator $\mathscr{B}$ is linear, on the ground of (7) and (8) it is not difficult to verify the linearity of $\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \mathscr{B}, X^2]$ and $\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \tilde{\mathscr{B}}, X^2]$. Equalities (7) and (8) are applicable also to the topologization of the GDS spaces $\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \mathscr{B}, X^2]$ and $\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \tilde{\mathscr{B}}, X^2]$.

Proposition 1. *Let $\mathbf{\Phi} = (\phi_{ki})$ be a double sequence of moduli and $\mathbf{p}^2 = (p_{ki})$ be a bounded sequence of positive numbers with $r = \max\{1, \sup_{k,i} p_{ki}\}$. Let $\mathscr{A} = (a_{mnki})$ be a non-negative infinite matrix and let $\mathscr{B} = (b_{kilj})$ be an infinite matrix of scalars. Suppose that $(X, |\cdot|)$ is a seminormed space and $\Lambda \subset s^2$ is a solid DS space which is topologized by an absolutely monotone F-seminorm g_Λ.*

a) *If the sequence of moduli $\mathbf{\Phi}^{\mathbf{p}^2/r}$ satisfies the condition* (M5'), *then the GDS space $\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \mathscr{B}, X^2]$ may be topologized by the F-seminorm*

$$h^{\mathbf{\Phi}, \mathbf{p}^2}_{\Lambda, \mathscr{A}, \mathscr{B}}(\mathbf{x}^2) = g_\Lambda\left(\mathscr{A}^{1/r}\left(\mathbf{\Phi}^{\mathbf{p}^2}\left(\mathscr{B}\mathbf{x}^2\right)\right)\right).$$

b) *If $(\Lambda[\mathscr{A}]^r, g^r_{\Lambda,\mathscr{A}})$ is an AK-space and the sequence of moduli $\mathbf{\Phi}^{\mathbf{p}^2/r}$ satisfies the conditions* (M7) *and* (M8), *then $h^{\mathbf{\Phi}, \mathbf{p}^2}_{\Lambda, \mathscr{A}, \mathscr{B}}$ is an F-seminorm on $\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \tilde{\mathscr{B}}, X^2]$.*

If, in a) *and* b), *g_Λ is an F-norm, X is normed, $\mathscr{A}$ is essentially positive, and*

$$\mathscr{B}\mathbf{x}^2 = 0 \implies \mathbf{x}^2 = 0, \tag{9}$$

then $h^{\mathbf{\Phi}, \mathbf{p}^2}_{\Lambda, \mathscr{A}, \mathscr{B}}$ is an F-norm on $\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \mathscr{B}, X^2]$ and $\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \tilde{\mathscr{B}}, X^2]$.

Proof. Since by (7) we have

$$\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \mathscr{B}, X^2] = \Lambda[A]^r(\mathbf{\Phi}^{\mathbf{p}^2/r}, T, X^2, Y^2),$$

with $T = \mathscr{B}$ and $Y = X$, statement a) follows from Theorem 1 because of

$$h^{\mathbf{\Phi}, \mathbf{p}^2}_{\Lambda, \mathscr{A}, \mathscr{B}}(\mathbf{x}^2) = g_\Lambda\left(\mathscr{A}^{1/r}\left(\mathbf{\Phi}^{\mathbf{p}^2/r}\left(\mathscr{B}\mathbf{x}^2\right)\right)^r\right) = g^r_{\Lambda,\mathscr{A}}\left(\mathbf{\Phi}^{\mathbf{p}^2/r}(\mathscr{B}\mathbf{x}^2)\right).$$

Analogously, we deduce statement b) from Theorem 2 in view of (8). □

Now, if Λ is one of the spaces $\tilde{\ell}^2_\infty$, $\tilde{c}^2_0$, c^2_0 and $bc^2_0 = c^2_0 \cap \tilde{\ell}^2_\infty$, then clearly

$$\left(\mathbf{u}^2\right)^{1/r} \in \Lambda \iff \mathbf{u}^2 \in \Lambda.$$

Thus $\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \mathscr{B}, X^2]$ coincides with the set

$$\Lambda[\mathscr{A}, \mathbf{\Phi}, \mathbf{p}^2, \mathscr{B}, X^2] = \left\{ \mathbf{x}^2 \in s^2_{\mathscr{B}}(X^2) : \mathscr{A}\left(\mathbf{\Phi}^{\mathbf{p}^2}\left(\mathscr{B}\mathbf{x}^2\right)\right) \in \Lambda \right\},$$

where

$$\mathscr{A}\left(\mathbf{\Phi}^{\mathbf{p}^2}\left(\mathscr{B}\mathbf{x}^2\right)\right) = \left(\sum_{k,i} a_{mnki}\left(\phi_{ki}\left(\left|\sum_{l,j} b_{kilj} x_{lj}\right|\right)\right)^{p_{ki}}\right)_{m,n\in\mathbb{N}}$$

and $\Lambda \in \{\tilde{\ell}^2_\infty, \tilde{c}^2_0, bc^2_0\}$. Hence Proposition 1 gives the following corollary.

Corollary 1. *Let $\mathbf{\Phi}$, $\mathbf{p}^2$, $\mathscr{A}$, $\mathscr{B}$, and X be the same as in Proposition* 1. *If $\Lambda \in \left\{\tilde{\ell}_\infty^2, \tilde{c}_0^2, bc_0^2\right\}$ with $g_{_\Lambda} = \|\cdot\|_\infty$, then the GDS space $\Lambda[\mathscr{A}, \mathbf{\Phi}, \mathbf{p}^2, \mathscr{B}, X^2]$ may be topologized by the F-seminorm*

$$h^{\mathbf{\Phi},\mathbf{p}^2}_{\infty,\mathscr{A},\mathscr{B}}(\mathbf{x}^2) = \sup_{m,n}\left(\sum_{k,i} a_{mnki}\left(\phi_{ki}\left(\left|\sum_{l,j} b_{kilj}x_{lj}\right|\right)\right)^{p_{ki}}\right)^{1/r}$$

whenever the sequence of moduli $\mathbf{\Phi}^{\mathbf{p}^2/r}$ satisfies the condition (M5'). *Thereby, if X is normed, $\mathscr{A}$ is essentially positive, and condition* (9) *holds, then $h^{\mathbf{\Phi},\mathbf{p}^2}_{\infty,\mathscr{A},\mathscr{B}}$ is an F-norm on $\Lambda[\mathscr{A}, \mathbf{\Phi}, \mathbf{p}^2, \mathscr{B}, X^2]$.*

The proof of Proposition 1 shows that in the case $\mathscr{B} = \mathscr{I}$ statements of Proposition 1 and Corollary 1 remain true if X^2 is replaced by $\mathbf{X}^2$. Moreover, condition (9) is automatically satisfied for $\mathscr{B} = \mathscr{I}$. Thus the following is true.

Proposition 2. *Let $\mathbf{\Phi}$, $\mathbf{p}^2$, $\mathscr{A}$, and $(\Lambda, g_{_\Lambda})$ be the same as in Proposition* 1. *Then the following statements hold.*
a) *The GDS space*

$$\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \mathbf{X}^2] = \left\{\mathbf{x}^2 \in s^2(\mathbf{X}^2) : \mathscr{A}^{1/r}\left(\mathbf{\Phi}^{\mathbf{p}^2}\left(\mathbf{x}^2\right)\right) \in \Lambda\right\}$$

may be topologized by the F-seminorm

$$h^{\mathbf{\Phi},\mathbf{p}^2}_{\Lambda,\mathscr{A}}(\mathbf{x}^2) = g_{_\Lambda}\left(\mathscr{A}^{1/r}\left(\mathbf{\Phi}^{\mathbf{p}^2}\left(\mathbf{x}^2\right)\right)\right)$$

whenever the sequence of moduli $\mathbf{\Phi}^{\mathbf{p}^2/r}$ satisfies the condition (M5').

b) *If $(\Lambda[\mathscr{A}]^r, g^r_{_{\Lambda,\mathscr{A}}})$ is an AK-space and the sequence of moduli $\mathbf{\Phi}^{\mathbf{p}^2/r}$ satisfies the conditions* (M7) *and* (M8), *then $h^{\mathbf{\Phi},\mathbf{p}^2}_{\Lambda,\mathscr{A}}$ is an F-seminorm on*

$$\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \tilde{\mathscr{I}}, \mathbf{X}^2] = \left\{\mathbf{x}^2 \in \tilde{s}^2(\mathbf{X}^2) : \mathscr{A}^{1/r}\mathbf{\Phi}^{\mathbf{p}^2}(\mathbf{x}^2) \in \Lambda\right\}.$$

If, in a) *and* b), *$g_{_\Lambda}$ is an F-norm, the spaces X_{ki} are normed and $\mathscr{A}$ is essentially positive, then $h^{\mathbf{\Phi},\mathbf{p}^2}_{\infty,\mathscr{A}}$ is an F-norm on $\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \mathbf{X}^2]$ and $\Lambda[\mathscr{A}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \tilde{\mathscr{I}}, \mathbf{X}^2]$.*

Corollary 2. *Let $\mathbf{\Phi}$, $\mathbf{p}^2$, and $\mathscr{A}$ be the same as in Proposition* 1. *If $\Lambda \in \left\{\tilde{\ell}_\infty^2, \tilde{c}_0^2, bc_0^2\right\}$ with $g_{_\Lambda} = \|\cdot\|_\infty$, then the GDS space $\Lambda[\mathscr{A}, \mathbf{\Phi}, \mathbf{p}^2, \mathbf{X}^2]$ may be topologized by the F-seminorm*

$$h^{\mathbf{\Phi},\mathbf{p}^2}_{\infty,\mathscr{A}}(\mathbf{x}^2) = \sup_{m,n}\left(\sum_{k,i} a_{mnki}\left(\phi_{ki}\left(\left|x_{ki}\right|_{ki}\right)\right)^{p_{ki}}\right)^{1/r}$$

whenever the sequence of moduli $\mathbf{\Phi}^{\mathbf{p}^2/r}$ satisfies the condition (M5'). *Thereby, if $g_{_\Lambda}$ is an F-norm in Λ, the spaces X_{ki} are normed and $\mathscr{A}$ is essentially positive, then $h^{\mathbf{\Phi},\mathbf{p}^2}_{\infty,\mathscr{A}}$ is an F-norm on $\Lambda[\mathscr{A}, \mathbf{\Phi}, \mathbf{p}^2, \mathbf{X}^2]$.*

Proposition 2 (see also Remark 2) generalizes and corrects a theorem of Basu and Srivastava (see [1, Theorem 3.2]). Savas and Patterson [15] consider the space

$$\Lambda[\mathscr{A}, \phi] = \left\{\mathbf{u}^2 \in s^2 : \left(\sum_{k,i} a_{mnki}\phi\left(|u_{ki}|\right)\right) \in \Lambda\right\}$$

in the case if ϕ is a modulus and $\Lambda \in \{\tilde{\ell}^2_\infty, c^2_0\}$. Because c^2 and c^2_o are not contained in $\tilde{\ell}^2_\infty$, Theorems 3.3 and 3.6 of [15] may not be true in general. Corollary 2 allows us to say that the spaces $\Lambda[\mathscr{A},\phi]$ with $\Lambda \in \{\tilde{\ell}^2_\infty, \tilde{c}^2_0, bc^2_0\}$ may be topologized by the F-seminorm

$$h^{\phi}_{\infty,\mathscr{A}}(\mathbf{u}^2) = \sup_{m,n} \sum_{k,i} a_{mnki}\phi\left(|u_{ki}|\right)$$

whenever ϕ satisfies the condition (M5°) (or the condition (M6°)).

Another special form of Proposition 1 is related to the modulus functions $\phi_{ki}(t) = t$ $(k,i \in \mathbb{N},\ t \in \mathbb{R}^+)$. In such case

$$\frac{\phi^{\mathbf{p}^2/r}_{ki}(ut)}{\phi^{\mathbf{p}^2/r}_{ki}(t)} = \frac{(ut)^{p_{ki}/r}}{t^{p_{ki}/r}} = u^{p_{ki}/r}$$

and thus, by Remark 1, (M5') holds if and only if $\inf_{k,i} p_{ki} > 0$. The condition $\inf_{k,i} p_{ki} > 0$ also guarantees that the sequence of moduli $\Phi^{\mathbf{p}^2/r}$ satisfies the conditions (M7) and (M8) if $\phi_{ki}(t) = t$. These facts permit us to formulate the following proposition and its corollary.

Proposition 3. *Let* $\mathbf{p}^2$, $\mathscr{A}$, $\mathscr{B}$, *and* (Λ, g_Λ) *be the same as in Proposition* 1. *If* $\inf_{k,i} p_{ki} > 0$, *then the following is true.*

a) *The GDS space*

$$\Lambda[\mathscr{A}^{1/r},\mathbf{p}^2,\mathscr{B},X^2] = \left\{\mathbf{x}^2 \in s^2_{\mathscr{B}}(X^2) : \mathscr{A}^{1/r}\left(\mathscr{B}\mathbf{x}^2\right)^{\mathbf{p}^2} \in \Lambda\right\},$$

where

$$\mathscr{A}^{1/r}\left(\mathscr{B}\mathbf{x}^2\right)^{\mathbf{p}^2} = \left(\left(\sum_{k,i} a_{mnki}\left(\Big|\sum_{l,j} b_{kilj}x_{lj}\Big|\right)^{p_{ki}}\right)^{1/r}\right)_{m,n\in\mathbb{N}},$$

may be topologized by the F-seminorm

$$h^{\mathbf{p}^2}_{\Lambda,\mathscr{A},\mathscr{B}}(\mathbf{x}^2) = g_\Lambda\left(\mathscr{A}^{1/r}\left(\mathscr{B}\mathbf{x}^2\right)^{\mathbf{p}^2}\right).$$

b) *If* $(\Lambda[\mathscr{A}]^r, g^r_{\Lambda,\mathscr{A}})$ *is an AK-space*, *then* $h^{\mathbf{p}^2}_{\Lambda,\mathscr{A},\mathscr{B}}$ *is an F-seminorm on*

$$\Lambda[\mathscr{A}^{1/r},\mathbf{p}^2,\tilde{\mathscr{B}},X^2] = \left\{\mathbf{x}^2 \in s^2_{\mathscr{B}}(X^2) : \mathscr{B}\mathbf{x}^2 \in \tilde{s}^2(X^2) \text{ and } \mathscr{A}^{1/r}\left(\mathscr{B}\mathbf{x}^2\right)^{\mathbf{p}^2} \in \Lambda\right\}.$$

If, *in* a) *and* b), g_Λ *is an F-norm*, *the space X is normed*, $\mathscr{A}$ *is essentially positive and condition* (9) *holds*, *then* $h^{\mathbf{p}^2}_{\infty,\mathscr{A},\mathscr{B}}$ *is an F-norm on* $\Lambda[\mathscr{A}^{1/r},\mathbf{p}^2,\mathscr{B},X^2]$ *and* $\Lambda[\mathscr{A}^{1/r},\mathbf{p}^2,\tilde{\mathscr{B}},X^2]$.

Moreover, *for* $\mathscr{B} = \mathscr{I}$ *all previous statements remain true with* $\mathbf{X}^2$ *instead of* X^2.

Corollary 3. *Let* $\mathbf{p}^2$, $\mathscr{A}$, $\mathscr{B}$, *and X be the same as in Proposition* 1. *Suppose that* $\inf_{k,i} p_{ki} > 0$ *and* $\Lambda \in \{\tilde{\ell}^2_\infty, \tilde{c}^2_0, bc^2_o\}$ *with* $g_\Lambda = \|\cdot\|_\infty$. *Then the GDS space* $\Lambda[\mathscr{A},\mathbf{p}^2,\mathscr{B},X^2]$ *may be topologized by the F-seminorm*

$$h^{\mathbf{p}^2}_{\infty,\mathscr{A},\mathscr{B}}(\mathbf{x}^2) = \sup_{m,n}\left(\sum_{k,i} a_{mnki}\Big|\sum_{l,j} b_{kilj}x_{lj}\Big|^{p_{ki}}\right)^{1/r}.$$

Thereby, *if X is normed*, $\mathscr{A}$ *is essentially positive and condition* (9) *holds*, *then* $h^{\mathbf{p}^2}_{\infty,\mathscr{A},\mathscr{B}}$ *is an F-norm on* $\Lambda[\mathscr{A},\mathbf{p}^2,\mathscr{B},X^2]$.

Corollary 3 generalizes some results from [3,4,17].

Finally, let $\mathscr{A} = \mathscr{I}$. Then $(\Lambda[\mathscr{A}]^r, g^r_{\Lambda,\mathscr{A}}) = (\Lambda, g_\Lambda)$ and Propositions 1 b) and 3 b) yield the following statements.

Corollary 4. *Let* $\mathbf{\Phi}$, $\mathbf{p}^2$, $\mathscr{B}$, *and* (Λ, g_Λ) *be the same as in Proposition* 1.

a) *If* (Λ, g_Λ) *is an AK-space and* $\mathbf{\Phi}^{\mathbf{p}^2/r}$ *satisfies* (M7) *and* (M8), *then the GDS space* $\Lambda[\mathscr{I}^{1/r}, \mathbf{\Phi}, \mathbf{p}^2, \tilde{\mathscr{B}}, X^2]$ *may be topologized by the F-seminorm* $h^{\Phi,\mathbf{p}^2}_{\Lambda,\mathscr{B}}(\mathbf{x}^2) = g_\Lambda\left(\mathbf{\Phi}^{\mathbf{p}^2/r}\left(\mathscr{B}\mathbf{x}^2\right)\right)$.

b) *If* (Λ, g_Λ) *is an AK-space,* $\phi_{ki}(t) = t$, $(k, i \in \mathbb{N})$, *and* $\inf_{k,i} p_{ki} > 0$, *then* $h^{\mathbf{p}^2}_{\Lambda,\mathscr{B}}(\mathbf{x}^2) = g_\Lambda\left(\mathscr{I}^{1/r}\left(\mathscr{B}\mathbf{x}^2\right)^{\mathbf{p}^2}\right)$ *is an F-seminorm on the GDS space* $\Lambda[\mathscr{I}^{1/r}, \mathbf{p}^2, \tilde{\mathscr{B}}, X^2]$.

Since $(\tilde{c}_0^2, \|\cdot\|_\infty)$ is an AK-space, Corollary 4 is applicable to $\Lambda = \tilde{c}_0^2$ with $\mathscr{I}$ instead of $\mathscr{I}^{1/r}$.

4. CONCLUSION

The topologization is an essential problem in the theory of various vector spaces, including theory of sequence spaces. It should be noted that the determination of F-seminorm or paranorm topologies for the double sequence spaces has not been studied as intensively as for the spaces of single sequences. We consider the topologization of a wide class of spaces of vector-valued double sequences which are defined by means of a solid F-seminormed space Λ of a double number sequences, a double sequence Φ of modulus functions, and a linear operator T. Our main theorems are applied in the case, where Λ is the strong summability domain of a non-negative 4-dimensional matrix $\mathscr{A}$ and the operator T is determined by an arbitrary 4-dimensional matrix $\mathscr{B}$. We also correct some inaccuracies of two previous papers.

ACKNOWLEDGEMENTS

Research of the first author was supported by the Estonian Targeted Financing Project SF0180039s08. Research of the second author was supported by the Estonian Targeted Financing Project SF0140011s09 and grant ETF9383 of the Estonian Science Foundation. The authors would like to thank the anonymous referees for some valuable comments and suggestions.

REFERENCES

1. Basu, A. and Srivastava, P. D. Generalized vector valued double sequence space using modulus function. *Tamkang J. Math.*, 2007, **4**, 347–366.
2. Ghosh, D. and Srivastava, P. D. On some vector valued sequence spaces defined using a modulus function. *Indian J. Pure Appl. Math.*, 1999, **30**, 819–826.
3. Gökhan, A. and Çolak, R. The double sequence spaces $c_2^P(p)$ and $c_2^{PB}(p)$. *Appl. Math. Comput.*, 2004, **157**, 491–501.
4. Gökhan, A. and Çolak, R. Double sequence space $\ell_2^\infty(p)$. *Appl. Math. Comput.*, 2005, **160**, 147–153.
5. Hazewinkel, M. (ed.). *Encyclopaedia of Mathematics, Vol. 1*. Kluwer Academic Publishers, Dordrecht, 1995.
6. Kamthan, P. K. and Gupta, M. *Sequence Spaces and Series*. Marcel Dekker, New York, 1981.
7. Kolk, E. F-seminormed sequence spaces defined by a sequence of modulus functions and strong summability. *Indian J. Pure Appl. Math.*, 1997, **28**, 1547–1566.
8. Kolk, E. Topologies in generalized Orlicz sequence spaces. *Filomat*, 2011, **25**, 191–211.
9. Maddox, I. J. Sequence spaces defined by a modulus. *Math. Proc. Cambridge Philos. Soc.*, 1986, **100**, 161–166.
10. Mölder, A. The topologization of sequence spaces defined by a matrix of moduli. *Proc. Estonian Acad. Sci. Phys. Math.*, 2004, **53**, 218–225.
11. Nakano, H. Concave modulars. *J. Math. Soc. Japan*, 1953, **5**, 29–49.
12. Pringsheim, A. Zur Theorie der zweifach unendlichen Zahlenfolgen. *Math. Ann.*, 1900, **53**, 289–321.
13. Raidjõe, A. *Sequence Spaces Defined by Modulus Functions and Superposition Operators*. Diss. Math. Univ. Tartu, **47**, Tartu, 2006.

14. Ruckle, W. H. FK spaces in which the sequence of coordinate vectors is bounded. *Canad. J. Math.*, 1973, **25**, 973–978.
15. Savas, E. and Patterson R. F. Double sequence spaces defined by a modulus. *Math. Slovaca*, 2011, **61**, 245–256.
16. Soomer, V. On the sequence space defined by a sequence of moduli and on the rate spaces. *Acta Comment. Univ. Tartu. Math.*, 1996, **1**, 71–74.
17. Tripathy, B. C. and Sarma, B. On some classes of difference double sequence spaces. *Fasc. Math.*, 2009, **41**, 135–142.
18. Zeller, K. and Beekmann, W. *Theorie der Limitierungsverfahren*. Springer, Berlin, 1970.

F-poolnormid moodulfunktsioonide abil defineeritud üldistatud topeltjadade ruumides

Enno Kolk ja Annemai Raidjõe

On leitud F-poolnormid ja F-normid selliste topeltjadade ruumides, mille elementideks on etteantud poolnormeeritud ruumide punktid ning mille teisendid kuuluvad reaalarvuliste elementidega topeltjadade ruumi Λ. Seejuures saadakse teisendatud jada lineaarse operaatori ja moodulfunktsioonide topeltjada rakendamise teel. Ruumi Λ kohta eeldatakse, et see on soliidne ja topologiseeritud absoluutselt monotoonse F-poolnormi või F-normi abil. Üldised tulemused leiavad rakendamist erijuhul, kui lineaarseks operaatoriks on 4-mõõtmelise maatriksiga määratud summeerimisoperaator ja ruum Λ on seotud 4-mõõtmelise mitte-negatiivse maatriksmenetluse tugeva summeeruvuse välja ning tugeva tõkestatuse väljaga.