Proceedings of the Estonian Academy of Sciences,
2014, **63**, 1, 2–7
doi: 10.3176/proc.2014.1.02
Available online at www.eap.ee/proceedings

MATHEMATICS

Two remarks on diameter 2 properties

Rainis Haller* and Johann Langemets

Institute of Mathematics, University of Tartu, J. Liivi 2, 50409 Tartu, Estonia; johann.langemets@ut.ee

Received 8 May 2013, revised 16 September 2013, accepted 23 September 2013, available online 14 March 2014

Abstract. A Banach space is said to have the diameter 2 property if the diameter of every nonempty relatively weakly open subset of its unit ball equals 2. In a paper by Abrahamsen, Lima, and Nygaard (Remarks on diameter 2 properties. *J. Conv. Anal.*, 2013, **20**, 439–452), the strong diameter 2 property is introduced and studied. This is the property that the diameter of every convex combination of slices of its unit ball equals 2. It is known that the diameter 2 property is stable by taking ℓ_p-sums for $1 \leq p \leq \infty$. We show the absence of the strong diameter 2 property on ℓ_p-sums of Banach spaces when $1 < p < \infty$. This confirms the conjecture of Abrahamsen, Lima, and Nygaard that the diameter 2 property and the strong diameter 2 property are different. We also show that the strong diameter 2 property carries over to the whole space from a non-zero M-ideal.

Key words: diameter 2 property, slice, relatively weakly open set.

1. INTRODUCTION

All Banach spaces considered in this note are over the real field. For a Banach space X, its dual space is denoted by X^*, B_X is the closed unit ball of X, and S_X stands for the unit sphere of X. By a *slice* of B_X we mean a set of the form

$$S(x^*, \alpha) = \{x \in B_X : x^*(x) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$.

Nygaard and Werner [10] showed that in every infinite-dimensional uniform algebra, every nonempty relatively weakly open subset of its closed unit ball has diameter 2. If a Banach space satisfies this condition, then it is said to have the *diameter* 2 *property* (see, e.g., [1,3,5]).

In addition to the diameter 2 property, Abrahamsen, Lima, and Nygaard [1] consider two other formally different diameter 2 properties – the local diameter 2 property and the strong diameter 2 property.

According to the terminology in [1], a Banach space X has the *local diameter* 2 *property* if every slice of B_X has diameter 2; and X has the *strong diameter* 2 *property* if every convex combination of slices of B_X has diameter 2, i.e., the diameter of $\sum_{i=1}^{n} \lambda_i S_i$ is 2, whenever $n \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_n \geq 0$, with $\sum_{i=1}^{n} \lambda_i = 1$, and $S_1, \ldots, S_n$ are slices of B_X.

The diameter 2 property clearly implies the local diameter 2 property. The strong diameter 2 property implies the diameter 2 property. This follows directly from Bourgain's lemma ([6, Lemma II.1 p. 26]), which asserts that every nonempty relatively weakly open subset of B_X contains some convex combination of slices.

* Corresponding author, rainis.haller@ut.ee
In this note, we summarize some main results on diameter 2 properties obtained in the Master's Thesis of the second named author. The thesis was defended at the University of Tartu in June 2012.

It is conjectured in [1] that these three diameter 2 properties are different. In Section 2, we will show that there exist Banach spaces with the diameter 2 property but without the strong diameter 2 property. In fact, we prove that the strong diameter 2 property is never stable by taking the ℓ_p-sum for $1 < p < \infty$ (cf. Theorem 1). On the other hand, the diameter 2 property is stable under ℓ_p-sums (see [1, Theorem 3.2]).

The papers [1] and [9] inspired us to consider diameter 2 properties in the context of M-ideals. Section 3 is the result of that study. We show that all three diameter 2 properties carry over to the whole space from a non-zero M-ideal. This generalizes Theorem 3.2 (the case of $p = \infty$) and Proposition 4.6 from [1].

2. STRONG DIAMETER 2 PROPERTY IS NEVER STABLE UNDER ℓ_p-SUMS

Perhaps the most surprising result in [1] is that the local diameter 2 property and the diameter 2 property are stable by taking ℓ_p-sums for $1 < p < \infty$ (see [1, Theorem 3.2]). The same result is true, and even easier also, for $p = 1$ and $p = \infty$. For $p = \infty$, the diameter 2 case was obtained by López Pérez ([9, Lemma 2.1], see also [4, Lemma 2.2]).

One of the questions asked in [1] was whether the strong diameter 2 property is also stable under ℓ_p-sums (see ([1, Question (c)]). The answer was known for $p = 1$ and for $p = \infty$:

- If the Banach spaces X and Y have the strong diameter 2 property, then $X \oplus_1 Y$ has the strong diameter 2 property (see [1, Theorem 2.7 (iii)]). This result is essentially due to Becerra Guerrero and López Pérez in [4, proof of Lemma 2.1 (ii)].
- If a Banach space X has the strong diameter 2 property, then $X \oplus_\infty Y$ has the strong diameter 2 property for any Banach space Y ([1, Proposition 4.6]). We will generalize the last result in Proposition 3.

The following is our main result. It provides an answer, in the negative, to Question (c) in [1]. Moreover, it confirms the conjecture in [1] that the diameter 2 property and the strong diameter 2 property are different.

Theorem 1. *Let X and Y be nontrivial Banach spaces and let $1 < p < \infty$. The Banach space $Z = X \oplus_p Y$ fails the strong diameter 2 property.*

Remark.
(1) Theorem 1 is a joint result with Märt Põldvere.
(2) Theorem 1 was obtained independently by María Acosta, Julio Becerra Guerrero, and Ginés López Pérez; it is included in [2, Theorem 3.2].
(3) Eve Oja has presented another proof of Theorem 1 ([8]).

To prove Theorem 1, we will need the following elementary lemma.

Lemma 2. *Let $1 < p < \infty$ and let q be such that $1/p + 1/q = 1$. If $z^* = (x^*, y^*)$ is an element in $S_{Z^*} = S_{X^* \oplus_q Y^*}$, then for every $\varepsilon > 0$ there exists $\alpha > 0$ such that*

$$\|(\|x\|, \|y\|) - (\|x^*\|^{q-1}, \|y^*\|^{q-1})\|_p < \varepsilon,$$

whenever $z = (x, y)$ is an element in $S(z^*, \alpha)$.

Proof. Note that if $z = (x, y)$ is an element in $S(z^*, \alpha)$, then $(\|x\|, \|y\|)$ and $(\|x^*\|^{q-1}, \|y^*\|^{q-1})$ are both elements of the slice $S((\|x^*\|, \|y^*\|), \alpha)$ of $B_{\ell_p^2}$. Obviously, when α tends to 0, then $\operatorname{diam}(S((\|x^*\|, \|y^*\|), \alpha))$ tends to 0 as well. This proves the result. □

Proof of Theorem 1. In fact, we will show a stronger statement: For every $\lambda \in (0, 1)$, there exists $\alpha, \beta > 0$ and $z^*, \tilde{z}^* \in S_{Z^*}$ such that

$$\lambda S(z^*, \alpha) + (1 - \lambda) S(\tilde{z}^*, \alpha) \subset (1 - \beta) B_Z.$$

Let $x^* \in S_{X^*}$ and $y^* \in S_{Y^*}$. We take $z^* = (x^*, 0)$ and $\tilde{z}^* = (0, y^*)$. Then z^* and $\tilde{z}^*$ are elements in S_{Z^*}. Fix $\lambda \in (0, 1)$. Let

$$\varepsilon = 1 - \left(\lambda^p + (1 - \lambda)^p\right)^{1/p}.$$

Clearly, $\varepsilon > 0$. By Lemma 2, there exists $\alpha > 0$ such that

$$\begin{aligned}\Big(\big(\lambda\,\|x\| + (1-\lambda)\,\|\tilde{x}\|\big)^p + \big(\lambda\,\|y\| + (1-\lambda)\,\|\tilde{y}\|\,\big)^p\Big)^{1/p}&\\ \le \Big(\big(\lambda\cdot 1 + (1-\lambda)\cdot 0\big)^p + \big(\lambda\cdot 0 + (1-\lambda)\cdot 1\big)^p\Big)^{1/p} + \frac{\varepsilon}{2}&\\ = \big(\lambda^p + (1-\lambda)^p\big)^{1/p} + \frac{\varepsilon}{2} = 1 - \frac{\varepsilon}{2},&\end{aligned}$$

whenever $z = (x,y) \in S(z^*, \alpha)$ and $\tilde{z} = (\tilde{x}, \tilde{y}) \in S(\tilde{z}^*, \alpha)$.

One may take $\beta = \varepsilon/2$. Indeed, for $z = (x,y) \in S(z^*, \alpha)$ and $\tilde{z} = (\tilde{x}, \tilde{y}) \in S(\tilde{z}^*, \alpha)$, we now have

$$\begin{aligned}\|\lambda z + (1-\lambda)\tilde{z}\| &= \big(\,\|\lambda x + (1-\lambda)\tilde{x}\|^p + \|\lambda y + (1-\lambda)\tilde{y}\|^p\big)^{1/p}\\ &\le \Big(\big(\lambda\,\|x\| + (1-\lambda)\,\|\tilde{x}\|\,\big)^p + \big(\lambda\,\|y\| + (1-\lambda)\,\|\tilde{y}\|\,\big)^p\Big)^{1/p}\\ &\le 1 - \frac{\varepsilon}{2}.\end{aligned}$$

□

3. DIAMETER 2 PROPERTIES CARRY OVER TO THE WHOLE SPACE FROM A NON-ZERO M-IDEAL

We denote the *annihilator* of a subspace Y of a Banach space X by

$$Y^\perp = \{x^* \in X^* \colon x^*(y) = 0 \quad \text{for all } y \in Y\}.$$

According to the terminology in [7], a closed subspace Y of a Banach space X is called an *M-ideal* if there exists a norm-1 projection P on X^* with $\ker P = Y^\perp$ and

$$\|x^*\| = \|Px^*\| + \|x^* - Px^*\| \qquad \text{for all } x^* \in X^*.$$

Relations between M-ideal structure and the diameter 2 property were first considered in [9]. There it is proved that if a proper subspace Y of X is an M-ideal in X and the range of the corresponding projection is 1-norming, then both Y and X have the diameter 2 property (see [9, Theorem 2.4]). In [1, Theorem 4.10] it is shown that, under the same assumptions, one can conclude that both Y and X have even the strong diameter 2 property. An immediate corollary of this is that if a nonreflexive Banach space X is an M-ideal in its bidual, then both X and X^{**} have the strong diameter 2 property.

One cannot omit the assumption that the range of the corresponding projection is 1-norming. To see an example of this, let Y be any Banach space and let $X = Y \oplus_\infty c_0$. Then, by [1, Proposition 4.6] (or Proposition 3 below), X has the strong diameter 2 property and Y is an M-ideal in X.

In the following we will show that if a non-zero M-ideal Y has some diameter 2 property, then X has the same diameter 2 property without the assumption that the range of the projection is 1-norming. This, at the same time, generalizes Theorem 3.2 (the case of $p = \infty$) and the above-mentioned Proposition 4.6 of [1].

Proposition 3. *Let X be a Banach space and let Y be a proper closed subspace of X. Assume that Y is an M-ideal in X. If Y has the strong diameter 2 property, then X has the strong diameter 2 property.*

Proof. Let $\sum_{i=1}^n \lambda_i S(x_i^*,\alpha_i)$ be a convex combination of slices of B_X, where $n\in\mathbb{N}$, and $\lambda_1,\ldots,\lambda_n\geq 0$ such that $\sum_{i=1}^n \lambda_i=1$. Let $\varepsilon>0$ be such that $\varepsilon<\min\{\alpha_1,\ldots,\alpha_n\}/3$.

We will show the existence of $x_1^1,\ldots,x_n^1,x_1^2,\ldots,x_n^2\in B_X$ such that $x_i^k\in S(x_i^*,\alpha_i)$ for every $i=1,\ldots,n$, $k=1,2$, and

$$\left\|\sum_{i=1}^n \lambda_i(x_i^1-x_i^2)\right\|>\frac{2-\varepsilon}{1+\varepsilon}.$$

Denote by P the M-ideal projection on X^* with $\ker P=Y^\perp$. For every $i=1,\ldots,n$, we take

$$y_i^*=\frac{Px_i^*}{\|Px_i^*\|}\qquad\text{and}\qquad \beta_i=\frac{\varepsilon-\varepsilon\,\|Px_i^*\|+\varepsilon^2}{\|Px_i^*\|}.$$

Note that, if $Px_i^*\neq 0$, then $\beta_i>0$. If $Px_i^*=0$, we can take $y_i^*\in S_{Y^*}$ and $\beta_i>0$ to be arbitrary. Observe that $\sum_{i=1}^n \lambda_i S(y_i^*,\beta_i)$ is a convex combination of slices of B_Y. Since Y has the strong diameter 2 property, we can find $y_1^1,\ldots,y_n^1$ and $y_1^2,\ldots,y_n^2$ in B_Y such that

$$Px_i^*(y_i^k)>(\|Px_i^*\|-\varepsilon)(1+\varepsilon),\qquad k=1,2,\quad i=1,\ldots,n,$$

and

$$\left\|\sum_{i=1}^n \lambda_i(y_i^1-y_i^2)\right\|>2-\varepsilon.$$

There are $x_1,\ldots,x_n\in B_X$ such that

$$(x_i^*-Px_i^*)(x_i)>(\|x_i^*-Px_i^*\|-\varepsilon)(1+\varepsilon)$$

for every $i=1,\ldots,n$.

Since Y is an M-ideal in X, then by [11, Proposition 2.3], we can, for every $i=1,\ldots,n$, choose $z_i\in B_Y$ such that

$$\left\|y_i^k+x_i-z_i\right\|<1+\varepsilon,\qquad k=1,2,$$

and

$$|Px_i^*(x_i-z_i)|<\varepsilon.$$

We take

$$x_i^k=\frac{y_i^k+x_i-z_i}{1+\varepsilon},\qquad k=1,2,\quad i=1,\ldots,n.$$

Now, for every $i=1,\ldots,n$, for every $k=1,2$, x_i^k is an element in $S(x_i^*,\alpha_i)$, because

$$\begin{aligned}x_i^*(x_i^k)&=\frac{x_i^*(y_i^k+x_i-z_i)}{1+\varepsilon}\\&=\frac{Px_i^*(y_i^k)+(x_i^*-Px_i^*)(x_i)+Px_i^*(x_i-z_i)}{1+\varepsilon}\\&>\|Px_i^*\|-\varepsilon+\|x_i^*-Px_i^*\|-\varepsilon-\varepsilon\\&=\|x_i^*\|-3\varepsilon>1-\alpha_i.\end{aligned}$$

Finally, observe that

$$\left\|\sum_{i=1}^n \lambda_i(x_i^1-x_i^2)\right\|=\frac{1}{1+\varepsilon}\left\|\sum_{i=1}^n \lambda_i(y_i^1-y_i^2)\right\|>\frac{2-\varepsilon}{1+\varepsilon}.$$ □

We conclude our study with the local diameter 2 and the diameter 2 versions of Proposition 3.

Proposition 4. *Let X be a Banach space and let Y be a proper closed subspace of X. Assume that Y is an M-ideal in X. If Y has the local diameter 2 property, then X has the local diameter 2 property.*

Proof. Take $n = 1$ in the proof of Proposition 3. $\square$

The next result is obtained in the proof of [9, Theorem 2.4], but not stated explicitly. We will give a direct proof of this result.

Proposition 5. *Let X be a Banach space and let Y be a proper closed subspace of X. Assume that Y is an M-ideal in X. If Y has the diameter 2 property, then X has the diameter 2 property.*

Proof. The proof is similar to the proof of Proposition 3.

Let U be a nonempty relatively weakly open subset of B_X containing an element x_0. We may assume that

$$\{x \in B_X : \ |x_i^*(x - x_0)| < \gamma, \quad i = 1, \dots, n\} \subset U,$$

for some $n \in \mathbb{N}$, $x_1^*, \dots, x_n^* \in S_{X^*}$, and $\gamma > 0$.

Denote by P the M-ideal projection on X^* with $\ker P = Y^\perp$, and let $\delta = \max\{\|Px_i^*\| : i = 1, \dots, n\}$. Let $\varepsilon > 0$ be such that $\varepsilon(4 + \delta) < \gamma$. We will show the existence of elements x and $\tilde{x}$ in U such that

$$\|x - \tilde{x}\| > \frac{2 - \varepsilon}{1 + \varepsilon}.$$

Since B_Y is dense in B_X in the weak topology $\sigma(X, \operatorname{ran} P)$, we can find an element $y_0 \in B_Y$ such that

$$|Px_i^*(x_0 - y_0)| < \varepsilon$$

for every $i = 1, \dots, n$. Consider the set

$$V = \{y \in B_Y : \ |Px_i^*(y - y_0)| < \varepsilon(\delta + 1), \quad i = 1, \dots, n\}.$$

Clearly V is a nonempty relatively weakly open subset of B_Y. By the assumption, there are $y_1, y_2 \in V$ with $\|y_1 - y_2\| > 2 - \varepsilon$.

Since Y is an M-ideal in X, by [11, Proposition 2.3], there is an element $z_0 \in B_Y$ such that

$$\|y_k + x_0 - z_0\| < 1 + \varepsilon, \qquad k = 1, 2,$$

and

$$|Px_i^*(x_0 - z_0)| < \varepsilon$$

for every $i = 1, \dots, n$.

We take

$$x_1 = \frac{y_1 + x_0 - z_0}{1 + \varepsilon} \qquad \text{and} \qquad x_2 = \frac{y_2 + x_0 - z_0}{1 + \varepsilon}.$$

Now, for every $i = 1, \dots, n$, we have

$$\begin{aligned} |x_i^*(x_1 - x_0)| &= \frac{1}{1 + \varepsilon} \, |x_i^*(y_1 - \varepsilon x_0 - z_0) \pm Px_i^*(x_0) \pm Px_i^*(y_0)| \\ &\leq \frac{1}{1 + \varepsilon} \Big(|Px_i^*(y_1 - y_0)| + |Px_i^*(x_0 - z_0)| + \varepsilon \, |x_i^*(x_0)| + |Px_i^*(y_0 - x_0)| \Big) \\ &< \frac{1}{1 + \varepsilon} (\varepsilon\delta + 4\varepsilon) < \gamma. \end{aligned}$$

Thus, $x_1 \in U$. Similarly one can show that $x_2 \in U$. Finally, observe that

$$\|x_1 - x_2\| = \frac{1}{1 + \varepsilon} \, \|y_1 - y_2\| > \frac{2 - \varepsilon}{1 + \varepsilon}. \qquad \square$$

ACKNOWLEDGEMENTS

We would like to express our gratitude to Märt Põldvere and Olav Nygaard for fruitful conversations about the subject of this paper. We are grateful to María Acosta, Julio Becerra Guerrero, and Ginés López Pérez for sending us their preprint [2], where the difference of the strong diameter 2 property and the diameter 2 property is obtained independently. Our recognition in [2] is very much appreciated. We are thankful to Eve Oja for useful comments concerning the presentation of this note and to the referees for helpful comments that improved our paper and for pointing out errors in our manuscript. This research was supported by Estonian Targeted Financing Project SF0180039s08 and by Estonian Science Foundation Grant 8976.

REFERENCES

1. Abrahamsen, T., Lima, V., and Nygaard, O. Remarks on diameter 2 properties. *J. Conv. Anal.*, 2013, **20**, 439–452.
2. Acosta, M. D., Becerra Guerrero, J., and López Pérez, G. Stability results of diameter two properties. *J. Conv. Anal.* (to appear).
3. Acosta, M. D., Becerra Guerrero, J., and Rodríguez Palacios, A. Weakly open sets in the unit ball of the projective tensor product of Banach spaces. *J. Math. Anal. Appl.*, 2011, **383**, 461–473.
4. Becerra Guerrero, J. and López Pérez, G. Relatively weakly open subsets of the unit ball in functions spaces. *J. Math. Anal. Appl.*, 2006, **315**, 544–554.
5. Becerra Guerrero, J. and Rodríguez Palacios, A. Relatively weakly open sets in closed balls of Banach spaces, and the centralizer. *Math. Z.*, 2009, **262**, 557–570.
6. Ghoussoub, N., Godefroy, G., Maurey, B., and Schachermayer, W. Some topological and geometrical structures in Banach spaces. *Mem. Amer. Math. Soc.*, 1987, **378**, 1–116.
7. Harmand, P., Werner, D., and Werner, W. *M-Ideals in Banach Spaces and Banach Algebras*. Lecture Notes in Mathematics, vol. 1547, Springer, Berlin, 1993.
8. Oja, E. Sums of slices in direct sums of Banach spaces. *Proc. Estonian Acad. Sci.*, 2014, **63**, 8–10.
9. López Pérez, G. The big slice phenomena in M-embedded and L-embedded spaces. *Proc. Amer. Math. Soc.*, 2005, **134**, 273–282.
10. Nygaard, O. and Werner, D. Slices in the unit ball of a uniform algebra. *Arch. Math.*, 2001, **76**, 441–444.
11. Werner, D. M-ideals and the “basic inequality”. *J. Approx. Theory*, 1994, **76**, 21–30.

Kaks märkust diameeter-2 omaduste kohta

Rainis Haller ja Johann Langemets

On tõestatud, et artiklis [1] vaadeldud diameeter-2 omadus ja tugev diameeter-2 omadus on erinevad. On näidatud, kuidas diameeter-2 omadused kanduvad M-ideaalilt kogu ruumile.