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CONTROL SYSTEMS

Observable space of the nonlinear control system on a homogeneous time scale

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Abstract. The observability property of the nonlinear system, defined on a homogeneous time scale, is studied. The observability condition is provided through the notion of the observable space. Moreover, the observability filtration and observability indices are defined and the decomposition of the system into observable/unobservable subsystems is considered.

Key words: nonlinear control system, time scale, observability, observable space.

1. INTRODUCTION

The theory of dynamical systems on time scales is a new and popular research area. From a modelling point of view, dynamical systems on time scales incorporate both the continuous- and discrete-time systems as special cases, allowing us to unify the study and consider the classical results as special cases of the new theory. However, it is important to note that the discrete-time model in the time scale formalism is given in terms of the difference operator, and not in terms of the more conventional shift operator as, for example, in [1–3,13]. The difference-based models, often referred to as delta-domain models, are not completely new for the description of discrete-time systems. They have been promoted during the last 20 years as the models closely linked to continuous-time systems, being less sensitive to round-off errors at higher sampling rates [12,20].

The properties (including observability) of linear systems, defined on time scales, were studied, for instance, in [5] and [11]. In [4] the algebraic formalism in terms of differential one-forms has been developed for the study of nonlinear control systems defined on homogeneous time scales and used later to study different problems like transfer equivalence, irreducibility, reduction, and realization of nonlinear input-output equations [7,17,18]. The formalism constructs the vector space of differential one-forms, defined over the differential field of meromorphic functions, associated with the control system. In the present paper we apply this formalism to define and construct the observable space for the nonlinear control system on a homogeneous time scale and define the observability indices of the system. Moreover, we provide the necessary and sufficient condition to check the single-experiment observability[1] of the system using

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[1] The multi-experiment observability of nonlinear control systems, defined on time scales, was studied in [22].

the notion of the observable space. Finally, we discuss the possibility of decomposing the system into observable/unobservable subsystems.

The paper is organized as follows. Preliminary information about the time scale calculus and algebraic framework is given in Section 2. The notions of observability, observability filtration, observable space, and observability indices are provided in Section 3. In Section 4 the decomposition of the system into observable/unobservable subsystems is studied. Section 5 provides brief conclusions.

2. PRELIMINARIES

2.1. Time scale calculus

For a general introduction to the time scale calculus see [6]. Here we recall only those notions and facts that we need in this paper, in particular, the concept of delta derivative for real function defined on a homogeneous time scale.

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases comprise the continuous-time case, $\mathbb{T}=\mathbb{R}$, and the discrete-time cases, $\mathbb{T}=\mathbb{Z}$ and $\mathbb{T}=\tau\mathbb{Z}$ for $\tau>0$. We assume that $\mathbb{T}$ is a topological space with the topology induced by $\mathbb{R}$. In the definition of the delta derivative, the so-called forward jump operator plays an important role. For $t\in\mathbb{T}$ the *forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$ is defined by

$$\sigma(t):=\inf\{s\in\mathbb{T}\mid\ s>t\},$$

while the *backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$ is defined by

$$\rho(t):=\sup\{s\in\mathbb{T}\mid\ s<t\}.$$

In this definition we set in addition $\sigma(\max\mathbb{T})=\max\mathbb{T}$ if there exists a finite $\max\mathbb{T}$. Obviously $\sigma(t)$ is in $\mathbb{T}$ when $t\in\mathbb{T}$. This is because of our assumption that $\mathbb{T}$ is a closed subset of $\mathbb{R}$. The *graininess functions* $\mu:\mathbb{T}\to[0,\infty)$ and $\nu:\mathbb{T}\to[0,\infty)$ are defined by $\mu(t):=\sigma(t)-t$ and $\nu(t):=t-\rho(t)$, respectively. A time scale $\mathbb{T}$ is called *homogeneous*[2] if $\mu=\nu\equiv\text{const}$. Let $\mathbb{T}^{\kappa}$ denote a truncated set consisting of $\mathbb{T}$ except for a possible maximal point such that $\rho(\max\mathbb{T})<\max\mathbb{T}$.

Definition 2.1. *Let $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^{\kappa}$. The* delta derivative *of f at t, denoted by $f^{\Delta}(t)$, is the real number (provided it exists) with the property that given any $\varepsilon>0$, there is a neighbourhood $U=(t-\delta,t+\delta)\cap\mathbb{T}$ (for some $\delta>0$) such that*

$$|(f(\sigma(t))-f(s))-f^{\Delta}(t)(\sigma(t)-s)|\leq\varepsilon|\sigma(t)-s|$$

for all $s\in U$. Moreover, we say that f is delta differentiable *on $\mathbb{T}^{\kappa}$, provided $f^{\Delta}(t)$ exists for all $t\in\mathbb{T}^{\kappa}$.*

Example 2.2.
- If $\mathbb{T}=\mathbb{R}$, then $\mu(t)\equiv 0$ and the delta derivative is the ordinary time derivative.
- If $\mathbb{T}=\tau\mathbb{Z}$, $\tau>0$, then $\mu(t)=\tau$ and $f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}=\frac{f(t+\tau)-f(t)}{\tau}$ is the difference operator.

For a function $f:\mathbb{T}\to\mathbb{R}$ one can define the 2nd delta derivative $f^{[2]}:=\left(f^{\Delta}\right)^{\Delta}:\mathbb{T}^{\kappa^2}\to\mathbb{R}$ provided that f^{Δ} is delta differentiable on $\mathbb{T}^{\kappa^2}:=(\mathbb{T}^{\kappa})^{\kappa}$. In a similar manner one defines higher-order delta derivatives $f^{[n]}:=\left(f^{[n-1]}\right)^{\Delta}:\mathbb{T}^{\kappa^n}\to\mathbb{R}$, where $\mathbb{T}^{\kappa^n}=\left(\mathbb{T}^{\kappa^{n-1}}\right)^{\kappa}$, $n\geq 1$. For notational convenience, denote $f^{[i\ldots n]}:=\left(f^{[i]},\ldots,f^{[n]}\right)$, for $0\leq i\leq n$ and $f^{[0]}:=f$.

[2] Although the closed interval $[a,b]$ is also an example of a homogeneous time scale, we restrict our consideration to infinite homogeneous time scales $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\tau\mathbb{Z}$ for $\tau>0$.

2.2. Algebraic framework

In this subsection we recall some notions and facts from [4], necessary for our study.

Consider a multi-input multi-output (MIMO) nonlinear control system, defined on the homogeneous time scale $\mathbb{T}$, and described by the state equations

$$\begin{aligned} x^{\Delta} &= f(x,u), \\ y &= h(x), \end{aligned} \tag{1}$$

where $x : \mathbb{T} \to \mathbb{X} \subset \mathbb{R}^n$ is an n-dimensional state vector, $u : \mathbb{T} \to \mathbb{U} \subset \mathbb{R}^m$ is an m-dimensional input vector, and $y : \mathbb{T} \to \mathbb{Y} \subset \mathbb{R}^p$ is a p-dimensional output vector. Moreover, $f : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ and $h : \mathbb{X} \to \mathbb{Y}$ are assumed to be real analytic functions.

Remark 2.3. Note that we are focusing neither on local nor global, but on the generic properties of the system, i.e. the properties that hold almost everywhere, except on a set of measure zero. Although the notion of generic property does not make sense, in general, for systems defined by C^∞ functions, the choice of analytic functions allows us to employ the generic approach. Morover, unlike the ring of C^∞ functions, the ring of analytic functions is integral domain, meaning that it can be embedded into its quotient field whose elements are meromorphic functions.

Assume that the map $(x,u) \mapsto \widetilde{f}(x,u) := x + \mu f(x,u)$ generically defines a submersion, i.e. generically

$$\operatorname{rank} \frac{\partial \widetilde{f}(x,u)}{\partial\,(x,u)} = n \tag{2}$$

holds. Assumption (2) is not restrictive, since it is a necessary condition for system accessibility [13] and is always satisfied in the case of $\mu \equiv 0$. Consider the infinite set of (independent) real indeterminates $\mathscr{C} := \{x_i,\ i = 1,\dots,n;\ u_\upsilon^{[k]},\ \upsilon = 1,\dots,m,\ k \geq 0\}$. Let $\mathscr{K}$ denote the field of meromorphic functions in a finite number of variables from the set $\mathscr{C}$. Thus for each $F \in \mathscr{K}$ there is $k \geq 0$ such that F depends on x and $u^{[0\dots k]}$. Let $\sigma_f : \mathscr{K} \to \mathscr{K}$ be the *forward shift operator* defined by

$$F^{\sigma_f}\left(x, u^{[0\dots k+1]}\right) := F\left(x + \mu f(x,u), u^{[0\dots k]} + \mu u^{[1\dots k+1]}\right).$$

Under the submersivity assumption, σ_f is injective endomorphism and so the operator σ_f is well defined on $\mathscr{K}$ (see [4]). Furthermore, define the operator $\Delta_f : \mathscr{K} \to \mathscr{K}$ by

$$F^{\Delta_f}\left(x, u^{[0\dots k+1]}\right) = \begin{cases} \dfrac{F^{\sigma_f}\left(x,u^{[0\dots k+1]}\right) - F\left(x,u^{[0\dots k]}\right)}{\tau} & \text{if } \mathbb{T} = \tau\mathbb{Z}, \tau > 0, \\ \dfrac{\partial F}{\partial x}\left(x,u^{[0\dots k]}\right) f(x,u) + \displaystyle\sum_{k\geq 0} \frac{\partial F}{\partial u^{[0\dots k]}}\left(x,u^{[0\dots k]}\right) u^{[1\dots k+1]} & \text{if } \mathbb{T} = \mathbb{R}. \end{cases}$$

Proposition 2.4. *Let $F : \mathscr{K} \to \mathscr{K}$, $G : \mathscr{K} \to \mathscr{K}$. The delta derivative satisfies the following properties*:
(i) $F^{\sigma_f} = F + \mu F^{\Delta_f}$,
(ii) $(\alpha F + \beta G)^{\Delta_f} = \alpha F^{\Delta_f} + \beta G^{\Delta_f}$, *for* $\alpha, \beta \in \mathbb{R}$,
(iii) $(FG)^{\Delta_f} = F^{\sigma_f} G^{\Delta_f} + F^{\Delta_f} G$ (*generalization of the Leibniz rule*),
(iv) *on a homogeneous time scale operators* Δ_f *and* σ_f *commute*, *i.e.*

$$\left(F^{\sigma_f}\right)^{\Delta_f} = \left(F^{\Delta_f}\right)^{\sigma_f}.$$

An operator Δ_f satisfying the rule (iii) of Proposition 2.4 is called a *σ_f-derivation* [9]. A commutative field endowed with a σ_f-derivation is called a *differential field*. The field $\mathscr{K}$ is endowed with a σ_f-differential structure determined by system (1), and there exists the differential overfield $\mathscr{K}^*$, called the *inversive closure* of $\mathscr{K}$. In [4] the construction of the inversive closure $\mathscr{K}^*$ for system (1) is given. The extension of σ_f to $\mathscr{K}^*$ is an automorphism [9].

Consider the infinite set of symbols $\mathrm{d}\mathscr{C} = \{\mathrm{d}x_i,\, i = 1,\dots,n;\, \mathrm{d}u_\upsilon^{[k]},\, \upsilon = 1,\dots,m,\, k \geq 0\}$ and denote by $\mathscr{E}$ the vector space over the field $\mathscr{K}^*$ spanned by the elements of $\mathrm{d}\mathscr{C}$, namely

$$\mathscr{E} = \mathrm{span}_{\mathscr{K}^*}\mathrm{d}\mathscr{C}.$$

Any element of $\mathscr{E}$ has the form

$$\omega = \sum_{i=1}^{n} A_i \mathrm{d}x_i + \sum_{k\geq 0}\sum_{\upsilon=1}^{m} B_{\upsilon k}\mathrm{d}u_\upsilon^{[k]},$$

where only a finite number of coefficients $B_{\upsilon k}$ are nonzero elements of $\mathscr{K}^*$. The elements of $\mathscr{E}$ will be called the *differential one-forms*.

Let us define the operator $\mathrm{d} : \mathscr{K}^* \to \mathscr{E}$ as follows:

$$\mathrm{d}F\left(x, u^{[0\dots k]}\right) := \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\left(x, u^{[0\dots k]}\right)\mathrm{d}x_i + \sum_{l=0}^{k}\sum_{\upsilon=1}^{m} \frac{\partial F}{\partial u_\upsilon^{[l]}}\left(x, u^{[0\dots k]}\right)\mathrm{d}u_\upsilon^{[l]}. \tag{3}$$

Let $\omega = \sum_i A_i \mathrm{d}\zeta_i$ be a one-form, where $A_i \in \mathscr{K}^*$ and $\zeta_i \in \mathscr{C}$. We define the operators $\Delta_f : \mathscr{E} \to \mathscr{E}$ and $\sigma_f : \mathscr{E} \to \mathscr{E}$ by

$$\omega^{\Delta_f} := \sum_i \left(A_i^{\Delta_f}\mathrm{d}\zeta_i + A_i^{\sigma_f}\mathrm{d}\left(\zeta_i^{\Delta_f}\right)\right) \tag{4}$$

and

$$\omega^{\sigma_f} := \sum_i A_i^{\sigma_f}\mathrm{d}\left(\zeta_i^{\sigma_f}\right).$$

Since $A_i^{\sigma_f} = A_i + \mu A_i^{\Delta_f}$,

$$\omega^{\Delta_f} = \sum_i \left(A_i^{\Delta_f}\mathrm{d}\zeta_i + \left(A_i + \mu A_i^{\Delta_f}\right)\mathrm{d}\left(\zeta_i^{\Delta_f}\right)\right).$$

One says that $\omega \in \mathscr{E}$ is an *exact* one-form if $\omega = \mathrm{d}F$ for some $F \in \mathscr{K}^*$. A one-form ω for which $\mathrm{d}\omega = 0$ is said to be *closed*. It is well known that exact forms are closed, while closed forms are only locally exact. Integrability of the subspace of one-forms may be checked by the Frobenius theorem below, where the symbol $\mathrm{d}\omega_i$ means the exterior derivative of one-form ω_i and $\wedge$ means the exterior or wedge product (for details see [8]).

Theorem 2.5. ([8]). *Let $\mathscr{V} = \mathrm{span}_{\mathscr{K}^*}\{\omega_1,\dots,\omega_r\}$ be a subspace of $\mathscr{E}$. The subspace $\mathscr{V}$ is integrable if and only if*

$$\mathrm{d}\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_r = 0$$

for any $i = 1,\dots,r$.

3. OBSERVABILITY AND OBSERVABLE SPACE

Frequently the observability rank condition is used to check whether the continuous-time nonlinear system is locally weakly observable [10,14]. This condition is sufficient for an arbitrary initial state and necessary for almost all initial states. Thus, we introduce the definition of observability for nonlinear systems, defined on homogeneous time scales, through the observability rank condition.

Definition 3.1. *System* (1) *is called* generically (single-experiment) observable *if the rank of the observability matrix is generically equal to n, i.e. if*

$$\mathrm{rank}_{\mathscr{K}^*}\left[\frac{\partial\left(h_1, h_1^{\Delta_f},\dots,h_1^{[n-1]},\dots,h_p, h_p^{\Delta_f},\dots,h_p^{[n-1]}\right)}{\partial x}\right] = n. \tag{5}$$

Observe that $h_\nu^{\sigma_f^k} := \left(h_\nu^{\sigma_f^{k-1}}\right)^{\sigma_f}$ for $k \geq 2$ and take into account that for $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$ the higher-order delta derivative can be computed explicitly as

$$h_\nu^{[i]} = \frac{1}{\tau^i}\sum_{k=0}^{i}(-1)^k C_i^k h_\nu^{\sigma_f^{i-k}}, \tag{6}$$

where C_i^k is the binomial coefficient, i.e. $C_i^k = \frac{i!}{(i-k)!k!}$.

Proposition 3.2. *For* $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$, *the following holds*:

$$\operatorname{rank}_{\mathscr{K}^*}\left[\frac{\partial\left(h_1, h_1^{\Delta_f}, \ldots, h_1^{[n-1]}, \ldots, h_p, h_p^{\Delta_f}, \ldots, h_p^{[n-1]}\right)^{\mathrm{T}}}{\partial x}\right]$$
$$= \operatorname{rank}_{\mathscr{K}^*}\left[\frac{\partial\left(h_1, h_1^{\sigma_f}, \ldots, h_1^{\sigma_f^{n-1}}, \ldots, h_p, h_p^{\sigma_f}, \ldots, h_p^{\sigma_f^{n-1}}\right)^{\mathrm{T}}}{\partial x}\right]. \tag{7}$$

Proof. Using (6), the arbitrary row of the left-hand side matrix in (7) may be rewritten as

$$\frac{\partial h_\nu^{[i]}}{\partial x} = \frac{1}{\tau^i}\sum_{k=0}^{i}(-1)^k C_i^k \cdot \frac{\partial h_\nu^{\sigma_f^{i-k}}}{\partial x}$$

for $\nu = 1, \ldots, p$ and $i = 1, \ldots, n-1$. Separating the first addend of the above sum yields

$$\frac{\partial h_\nu^{[i]}}{\partial x} = \frac{1}{\tau^i}\left(\frac{\partial h_\nu^{\sigma_f^i}}{\partial x} + \sum_{k=1}^{i}(-1)^k C_i^k \cdot \frac{\partial h_\nu^{\sigma_f^{i-k}}}{\partial x}\right).$$

Now the sum $\sum_{k=1}^{i}$ in the above equality is the linear combination of the previous rows of the matrix and therefore can be removed without changing the rank of the matrix. Since $\partial h_\nu^{\sigma_f^i}/\partial x$ is the row of the right-hand side matrix of (7) for $i = 1, \ldots, n-1$, the statement of the proposition holds. □

Remark 3.3. Since for $\mathbb{T} = \mathbb{R}$ the delta derivative coincides with the classical time derivative, the condition (5) is equivalent to the observability rank condition in [10]. By Proposition 3.2 in the discrete-time case the condition (5) is equivalent to the observability rank condition given in [16].

Although Definition 3.1 may be applied to check observability, it is easier to be done using a concept of observable space like in the continuous-time case [10]. Moreover, the observable space, if integrable, allows us to decompose the system into observable/unobservable subsystems. In the remaining part of this section we extend the concept of observable space to the case of (MIMO) systems, defined on homogeneous time scales, and, using the notion of observable space, provide the necessary and sufficient observability condition.

Given system (1), denote by $\mathscr{X}$, $\mathscr{Y}^k$, $\mathscr{Y}$, and $\mathscr{U}$ the following subspaces of differential one-forms:

$$\begin{aligned}
\mathscr{X} &:= \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x\},\\
\mathscr{Y}^k &:= \operatorname{span}_{\mathscr{K}^*}\left\{\mathrm{d}h_\nu^{[j]}, \nu = 0, \ldots, p, 0 \leq j \leq k\right\},\\
\mathscr{Y} &:= \operatorname{span}_{\mathscr{K}^*}\left\{\mathrm{d}h_\nu^{[j]}, \nu = 0, \ldots, p, j \geq 0\right\},\\
\mathscr{U} &:= \operatorname{span}_{\mathscr{K}^*}\left\{\mathrm{d}u_\upsilon^{[l]}, \upsilon = 1, \ldots, m, l \geq 0\right\}.
\end{aligned} \tag{8}$$

By analogy with [10], the finite chain of subspaces

$$0 \subset \mathscr{O}_0 \subset \mathscr{O}_1 \subset \cdots \subset \mathscr{O}_k \subset \cdots \subset \mathscr{O}_{k^*-1} = \mathscr{O}_{k^*} =: \mathscr{O}_\infty, \tag{9}$$

where

$$\mathscr{O}_k := \mathscr{X} \cap \left(\mathscr{Y}^k + \mathscr{U}\right), \tag{10}$$

is called the *observability filtration*. Denote by $\mathscr{O}_\infty$ the limit of the observability filtration. It is easy to see that

$$\mathscr{O}_\infty = \mathscr{X} \cap (\mathscr{Y} + \mathscr{U})$$

and analogously with [10] we call the subspace $\mathscr{O}_\infty$ of $\mathscr{X}$ the *observable space*[3] of system (1). The unobservable space of system (1), denoted by $\mathscr{X}_{\bar{O}}$, is defined as a subspace of $\mathscr{X}$, which satisfies

$$\mathscr{X}_{\bar{O}} \cong \mathscr{X} / \mathscr{O}_\infty, \quad \mathscr{X}_{\bar{O}} \oplus \mathscr{O}_\infty = \mathscr{X},$$

where $\mathscr{X} / \mathscr{O}_\infty$ denotes the factor-space.

From (8), taking into account (3) and using the linear transformations, one obtains

$$\mathscr{Y}^k + \mathscr{U} = \operatorname{span}_{\mathscr{K}^*}\left\{\frac{\partial h_\nu^{[j]}}{\partial x}\mathrm{d}x, \nu = 1,\ldots,p, 0 \le j \le k; \mathrm{d}u_\upsilon^{[l]}, \upsilon = 1,\ldots,m, l \ge 0\right\}.$$

Consequently, according to (10),

$$\mathscr{O}_k = \operatorname{span}_{\mathscr{K}^*}\left\{\frac{\partial h_\nu^{[j]}}{\partial x}\mathrm{d}x, \nu = 1,\ldots,p, 0 \le j \le k\right\}, \tag{11}$$

yielding

$$\mathscr{O}_\infty = \operatorname{span}_{\mathscr{K}^*}\left\{\frac{\partial h_\nu^{[j]}}{\partial x}\mathrm{d}x, \nu = 1,\ldots,p, j \ge 0\right\}.$$

Before studying the properties of the observable space we provide Lemma 3.4. Denote the one-forms which generate the observable space $\mathscr{O}_\infty$ as $\omega_{\nu,j} := \frac{\partial h_\nu^{[j]}}{\partial x}\mathrm{d}x$ for $\nu = 1,\ldots,p, j \ge 0$ and arrange them in the form of the following matrix:

$$\Omega = \begin{bmatrix} \omega_{1,0} & \omega_{1,1} & \omega_{1,2} & \cdots \\ \omega_{2,0} & \omega_{2,1} & \omega_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \\ \omega_{p,0} & \omega_{p,1} & \omega_{p,2} & \cdots \end{bmatrix}.$$

Also denote the arbitrary row of the above matrix by Ω_ν.

Lemma 3.4. *If Ω_ν contains the one-form $\omega_{\nu,i}$, which is a linear combination of the former one-forms $\omega_{\nu,0},\ldots,\omega_{\nu,i-1}$ from Ω_ν, then the next one-forms $\omega_{\nu,j}$ for $j > i$ can also be represented as a linear combination of the one-forms $\omega_{\nu,0},\ldots,\omega_{\nu,i-1}$.*

The proof of Lemma 3.4 is given in the Appendix.

The proposition below describes the property of the subspace $\mathscr{O}_\infty$.

Proposition 3.5.

$$\dim_{\mathscr{K}^*}\mathscr{O}_\infty = \operatorname{rank}_{\mathscr{K}^*}\left[\frac{\partial\left(h_1, h_1^{\Delta_f}, \ldots, h_1^{[n-1]}, \ldots, h_p, h_p^{\Delta_f}, \ldots, h_p^{[n-1]}\right)}{\partial x}\right].$$

[3] Note that $\mathscr{O}_\infty$ is in general not the observation space as in [23], associated with the concept of the multi-experiment observability.

Proof. Represent the observable space as

$$\mathscr{O}_\infty = \mathscr{O}_\infty^1 + \mathscr{O}_\infty^2 + \cdots + \mathscr{O}_\infty^p,$$

where $\mathscr{O}_\infty^\nu$ is generated by the elements of Ω_ν. Since $\mathscr{O}_\infty^\nu \subseteq \mathscr{O}_\infty \subseteq \mathscr{X}$ and, as a consequence, $\dim \mathscr{O}_\infty^\nu \leq \dim \mathscr{O}_\infty \leq \dim \mathscr{X} = n$, it is enough to use n independent differential one-forms $\omega_{\nu,j}$ to generate $\mathscr{O}_\infty^\nu$. Lemma 3.4 guarantees that the first n one-forms $\omega_{\nu,j}$, $0 \leq j \leq n-1$, span the subspace $\mathscr{O}_\infty^\nu$. Consequently,

$$\operatorname{span}_{\mathscr{K}^*}\left\{\frac{\partial h_\nu^{[j]}}{\partial x}\mathrm{d}x, \nu = 1,\ldots,p, j \geq 0\right\} = \operatorname{span}_{\mathscr{K}^*}\left\{\frac{\partial h_\nu^{[j]}}{\partial x}\mathrm{d}x, \nu = 1,\ldots,p, 0 \leq j \leq n-1\right\}.$$

Thus, the rows of the observability matrix

$$\left[\frac{\partial\left(h_1, h_1^{\Delta_f}, \ldots, h_1^{[n-1]}, \ldots, h_p, h_p^{\Delta_f}, \ldots, h_p^{[n-1]}\right)}{\partial x}\right] \tag{12}$$

with n columns can be regarded as the representation of the elements of the codistribution $\mathscr{O}_\infty$. Therefore, the number of linearly independent vectors of $\mathscr{O}_\infty$, i.e. $\dim_{\mathscr{K}^*} \mathscr{O}_\infty$, can be found as the rank of the matrix (12). □

The following theorem is a direct consequence of Definition 3.1 and Proposition 3.5 and provides the characterization of the observability of the system.

Theorem 3.6. *A system* (1) *is* (*single-experiment*) *observable if and only if* $\mathscr{O}_\infty = \mathscr{X}$.

Example 3.7. Consider the continuous-time model of unicycle [10] and its discrete-time approximation, based on the Euler sampling scheme, as a single model defined on the homogeneous time scale $\mathbb{T}$:

$$\begin{aligned} x_1^\Delta &= u_1 \cos x_3, \\ x_2^\Delta &= u_1 \sin x_3, \\ x_3^\Delta &= u_2, \\ y_1 &= x_1, \\ y_2 &= x_2. \end{aligned} \tag{13}$$

Using (11), the observability filtration (9) of system (13) may be computed as follows:

$$\begin{aligned} \mathscr{O}_0 &= \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x_1, \mathrm{d}x_2\}, \\ \mathscr{O}_\infty = \mathscr{O}_1 &= \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3\}. \end{aligned}$$

Since the observable space $\mathscr{O}_\infty = \mathscr{X}$, the system is observable. Alternatively, one may check that direct application of Definition 3.1 yields the same result but requires more computations:

$$\operatorname{rank}_{\mathscr{K}^*}\left[\frac{\partial\left(h_1, h_1^{\Delta_f}, h_1^{[2]}, h_2, h_2^{\Delta_f}, h_2^{[2]}\right)}{\partial x}\right] = \operatorname{rank}_{\mathscr{K}^*}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -u_1 \sin x_3 \\ 0 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & u_1 \cos x_3 \\ 0 & 0 & b \end{bmatrix} = 3,$$

where

$$a := \begin{cases} \dfrac{u_1 \sin x_3 - \left(u_1 + \tau u_1^{\Delta}\right) \sin\left(\tau u_2 + x_3\right)}{\tau} & \text{if } \mathbb{T} = \tau\mathbb{Z}, \tau > 0, \\ -u_1 u_2 \cos x_3 - \dot{u}_1 \sin x_3 & \text{if } \mathbb{T} = \mathbb{R}, \end{cases}$$

$$b := \begin{cases} \dfrac{-u_1 \cos x_3 + \left(u_1 + \tau u_1^{\Delta}\right) \cos\left(\tau u_2 + x_3\right)}{\tau} & \text{if } \mathbb{T} = \tau\mathbb{Z}, \tau > 0, \\ -u_1 u_2 \sin x_3 + \dot{u}_1 \cos x_3 & \text{if } \mathbb{T} = \mathbb{R}. \end{cases}$$

Given a system of the form (1), its observability filtration (9), like in the continuous-time case [10], defines a set of structural indices σ_j for $j = 1, \ldots, k^*$ by

$$\begin{aligned} \sigma_1 &= \dim_{\mathscr{K}^*} \mathscr{O}_0, \\ \sigma_j &= \dim_{\mathscr{K}^*} \left(\mathscr{O}_{j-1}/\mathscr{O}_{j-2}\right), \quad j = 2, \ldots, k^*. \end{aligned} \tag{14}$$

Another set of indices s_i, for $i = 1, \ldots, p$, being dual to the set $\left\{\sigma_j, j = 1, \ldots, k^*\right\}$, is defined by

$$s_i = \operatorname{card}\left\{\sigma_j \mid \sigma_j \geq i\right\}$$

and called the set of *observability indices* of system (1). The integer σ_j represents the number of observability indices s_i which are greater than or equal to j, and duality implies that $\sigma_j = \operatorname{card}\{s_i \mid s_i \geq j\}$.

Observability indices determine how many delta derivatives of the respective output components one needs to use for computation of the initial state x on the basis of the inputs and outputs and their delta derivatives. The following proposition describes the key property of the observability indices.

Proposition 3.8. *Given a system of the form* (1), *one has*

$$\dim_{\mathscr{K}^*} \mathscr{O}_\infty = s_1 + \cdots + s_p.$$

Proof. Note that $\dim_{\mathscr{K}^*} \left(\mathscr{O}_{j-1}/\mathscr{O}_{j-2}\right) = \dim_{\mathscr{K}^*} \mathscr{O}_{j-1} - \dim_{\mathscr{K}^*} \mathscr{O}_{j-2}$. Using (14), one can write

$$\sum_{j=1}^{k^*} \sigma_j = \sum_{j=1}^{k^*} \dim_{\mathscr{K}^*} \mathscr{O}_{j-1} - \sum_{j=2}^{k^*} \dim_{\mathscr{K}^*} \mathscr{O}_{j-2}. \tag{15}$$

Separating the last addend of the first sum in the right-hand side of (15), replacing in this sum index j by $j-1$, and taking into account that $\mathscr{O}_{k^*-1} = \mathscr{O}_\infty$, we obtain

$$\sum_{j=1}^{k^*} \sigma_j = \dim_{\mathscr{K}^*} \mathscr{O}_\infty + \sum_{j=2}^{k^*} \dim_{\mathscr{K}^*} \mathscr{O}_{j-2} - \sum_{j=2}^{k^*} \dim_{\mathscr{K}^*} \mathscr{O}_{j-2} = \dim_{\mathscr{K}^*} \mathscr{O}_\infty. \tag{16}$$

The relation between indices σ_j and s_i can be expressed by means of a $k^* \times p$ table, whose (j,i)th element is defined by ($j = 1, \ldots, k^*$ pointing to the row and $i = 1, \ldots, p$ to the column)

$$a_{j,i} = \begin{cases} 1, & 1 \leq i \leq \sigma_j, \\ 0, & (\sigma_j + 1) \leq i \leq p, \end{cases} = \begin{cases} 1, & 1 \leq j \leq s_i, \\ 0, & (s_i + 1) \leq j \leq k^*. \end{cases}$$

Thus, the indices σ_j and s_i are the sums of elements in the jth row and ith column, respectively, i.e.

$$\sigma_j = \sum_{i=1}^{p} a_{j,i}, \quad s_i = \sum_{j=1}^{k^*} a_{j,i}. \tag{17}$$

Taking (16) and (17) into account, one obtains

$$\sum_{i=1}^{p} s_i = \sum_{i=1}^{p} \sum_{j=1}^{k^*} a_{j,i} = \sum_{j=1}^{k^*} \sigma_j = \dim_{\mathscr{K}^*} \mathscr{O}_\infty,$$

which completes the proof. □

Example 3.9. (Continuation of Example 3.7). One has $\sigma_1 = 2$, $\sigma_2 = 1$ and so the observability indices are $s_1 = 2$, $s_2 = 1$. Taking delta derivatives of y_1 and y_2 up to the orders $s_1 - 1$ and $s_2 - 1$, respectively, we obtain $y_1 = x_1$, $y_1^\Delta = u_1 \cos x_3$, $y_2 = x_2$, yielding

$$\begin{aligned} x_1 &= y_1, \\ x_2 &= y_2, \\ x_3 &= \arccos \frac{y_1^\Delta}{u_2}. \end{aligned}$$

4. DECOMPOSITION

For certain applications it will be useful to have system representations in which observable and unobservable state variables can be explicitly distinguished. For a continuous-time nonlinear control system the decomposition into observable/unobservable subsystems has been carried out both via differential geometric [15,21] and linear algebraic methods [10] and is proved to be always possible. For example, in [10] the decomposition was first carried out for a linearized system defined in terms of one-forms, and then, it was proved that the observable subspace of differential one-forms is always (generically) integrable. Therefore, the observable subspace of one-forms can be (at least locally) spanned by exact one-forms whose integrals define the observable state coordinates. As demonstrated in [16], for the discrete-time nonlinear control systems described in terms of the shift operator σ_f the decomposition at the level of equations (state variables) is not always possible since the observable space of one-forms is not necessarily completely integrable. Moreover, the paper [19] provides a general subclass of systems with a non-integrable observable subspace.

The purpose of this section is to study the possibility of decomposing the nonlinear control system defined on a homogeneous time scale into observable and unobservable subsystems. Since the delta-domain model obtained via sampling [12] behaves similarly to the continuous-time system and at the limit, when the sampling frequency increases infinitely, approaches the continuous-time system, it was our working hypothesis that the delta-domain models are, in general, decomposable into observable/unobservable parts. This would mean that the respective observable space $\mathscr{O}_\infty$, as a space of differential one-forms, is completely integrable. In [10] the observable space $\mathscr{O}_\infty$ is proved to be integrable in the case of $\mu \equiv 0$ ($\mathbb{T} = \mathbb{R}$). Unfortunately, unlike the case $\mathbb{T} = \mathbb{R}$ for the case $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$, $\mathscr{O}_\infty$ is not necessarily integrable. We give a number of counterexamples.

Example 4.1. Consider the control system, defined on a homogeneous time scale:

$$\begin{aligned} x_1^\Delta &= x_3 + u x_3 - x_1, \\ x_2^\Delta &= u - x_2, \\ x_3^\Delta &= u x_1 - x_3 - x_2, \\ y &= x_3. \end{aligned} \tag{18}$$

By (9), for this system, $\mathscr{O}_\infty = \mathscr{O}_2 = \operatorname{span}_{\mathscr{K}^*}\{\mathrm{d}x_3, 2\mathrm{d}x_2 + (u^\Delta - \mu u^\Delta - 2u)\,\mathrm{d}x_1, \mathrm{d}x_2 - u\mathrm{d}x_1\}$. If $\mathbb{T} = \mathbb{R}$, then $\mu \equiv 0$ and obviously[4], $\mathscr{O}_\infty = \mathscr{X}$. If $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$, then $\mathscr{O}_\infty = \mathscr{X}$, except for the case $\mu =$

[4] Of course, for $\mu \equiv 0$ the result also follows from continuous-time theory [10].

$\tau = 1$ where $\mathscr{O}_\infty = \mathrm{span}_{\mathscr{K}^*}\{\mathrm{d}x_3, \mathrm{d}x_2 - u\mathrm{d}x_1\}$, which is a non-integrable subspace by Theorem 2.5, since $\mathrm{d}(\mathrm{d}x_2 - u\mathrm{d}x_1) \wedge \mathrm{d}x_3 \wedge (\mathrm{d}x_2 - u\mathrm{d}x_1) = \mathrm{d}u \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \neq 0$.

The next example demonstrates that the loss of integrability does not necessarily occur only at $\mu = 1$.

Example 4.2. Consider the system

$$\begin{aligned} x_1^\Delta &= x_2 - \frac{x_1}{3}, \\ x_2^\Delta &= ux_1 + x_3 - x_2, \\ x_3^\Delta &= \mathrm{e}^{u^2x_1 + ux_3} - \frac{x_3}{3}, \\ y &= x_2. \end{aligned} \tag{19}$$

The observable space of the system

$$\mathscr{O}_\infty = \mathscr{O}_2 = \mathrm{span}_{\mathscr{K}^*}\left\{\mathrm{d}x_2, u\mathrm{d}x_1 + \mathrm{d}x_3, \left(u^\Delta - \frac{\mu u^\Delta}{3}\right)\mathrm{d}x_1\right\}.$$

Like in the previous example, if $\mathbb{T} = \mathbb{R}$, then $\mu \equiv 0$ and $\mathscr{O}_\infty = \mathscr{X}$. If $\mathbb{T} = \tau\mathbb{Z}$, $\tau > 0$, then $\mathscr{O}_\infty = \mathscr{X}$, except for the case $\mu = \tau = 3$ where $\mathscr{O}_\infty = \mathrm{span}_{\mathscr{K}^*}\{\mathrm{d}x_2, u\mathrm{d}x_1 + \mathrm{d}x_3\}$, again non-integrable by the Frobenius theorem.

Finally, we provide an example of the system for which the observable space $\mathscr{O}_\infty$ is integrable for every choice of the value of μ.

Example 4.3. Consider the system

$$\begin{aligned} x_1^\Delta &= \tan(x_1 - x_2)u_1, \\ x_2^\Delta &= u_1 \tan(x_1 - x_2) - u_2 \cos^2(x_1 - x_2), \\ x_3^\Delta &= u_1, \\ y_1 &= x_3, \\ y_2 &= x_1 - x_2. \end{aligned} \tag{20}$$

The observable space $\mathscr{O}_\infty = \mathscr{O}_0 = \mathrm{span}_{\mathscr{K}^*}\{\mathrm{d}x_1 - \mathrm{d}x_2, \mathrm{d}x_3\}$ is obviously integrable by direct inspection.

To conclude, we conjecture that the observable space $\mathscr{O}_\infty$ is in general integrable, except for a few possible μ values where these values correspond to the sampling frequencies at which the state transition map of the sampled system is not reversible. The following example illustrates this conjecture.

Example 4.4. (Continuation of Examples 4.1–4.3). The state transition map of system (18) is

$$\begin{aligned} x_1^+ &= \mu(x_3 + ux_3 - x_1) + x_1, \\ x_2^+ &= \mu(u - x_2) + x_2, \\ x_3^+ &= \mu(ux_1 - x_3 - x_2) + x_3, \end{aligned} \tag{21}$$

where we use the notation $x^+ := x(t + \mu)$. In order to check the reversibility of the system, one needs to verify whether the Jacobian matrix $\partial \widetilde{f}(x,u)/\partial x$ is nonsingular. The Jacobian matrix of system (21) is

$$\frac{\partial \widetilde{f}(x,u)}{\partial x} = \begin{bmatrix} 1-\mu & 0 & \mu(1+u) \\ 0 & 1-\mu & 0 \\ \mu u & -\mu & 1-\mu \end{bmatrix}.$$

One can verify that the above matrix is singular for $\mu = 1$, implying that the state transition map (21) is not reversible at the sampling frequency equal to 1. Next, consider the state transition map of system (19), which reads as

$$
\begin{aligned}
x_1^+ &= \mu\left(x_2 - \frac{x_1}{3}\right) + x_1,\\
x_2^+ &= \mu\left(ux_1 + x_3 - x_2\right) + x_2,\\
x_3^+ &= \mu\left(\mathrm{e}^{u^2x_1+ux_3} - \frac{x_3}{3}\right) + x_3.
\end{aligned}
\tag{22}
$$

The Jacobian matrix of system (22), i.e.

$$
\frac{\partial \widetilde{f}(x,u)}{\partial x} = \begin{bmatrix} 1-\frac{\mu}{3} & \mu & 0\\ \mu u & 1-\mu & \mu\\ \mathrm{e}^{u(ux_1+x_3)}\mu u^2 & 0 & 1-\frac{\mu}{3}+\mathrm{e}^{u(ux_1+x_3)}\mu u \end{bmatrix},
$$

is singular for $\mu = 3$. Consequently, the state transition map (22) is not reversible at the sampling frequency equal to 3. Finally, the state transition map of system (20) is

$$
\begin{aligned}
x_1^+ &= \mu\tan(x_1-x_2)u_1 + x_1,\\
x_2^+ &= \mu\left(u_1\tan(x_1-x_2) - u_2\cos^2(x_1-x_2)\right) + x_2,\\
x_3^+ &= \mu u_1 + x_3
\end{aligned}
\tag{23}
$$

and its Jacobian matrix reads as

$$
\frac{\partial \widetilde{f}(x,u)}{\partial x} = \begin{bmatrix} 1+\frac{\mu u_1}{\cos^2(x_1-x_2)} & \frac{-\mu u_1}{\cos^2(x_1-x_2)} & 0\\ a & 1-a & 0\\ 0 & 0 & 1 \end{bmatrix},
$$

where $a := \mu\left(\frac{u_1}{\cos^2(x_1-x_2)} + u_2\sin\left(2\left(x_1-x_2\right)\right)\right)$. One can verify that the above matrix is nonsingular for any $\mu \equiv$ const, meaning that the state transition map (23) is reversible at any sampling frequency. To conclude, comparing the above result with those presented in Examples 4.1–4.3, one can observe the consistency of the sampling frequencies at which the state transition maps are not reversible and the values of μ for which the observable spaces $\mathscr{O}_\infty$ are not integrable. These examples support our conjecture.

If $\mathscr{O}_\infty$ is integrable, and therefore, has locally an exact basis $\{\mathrm{d}\zeta_1,\ldots,\mathrm{d}\zeta_r\}$, one can complete the set $\{\mathrm{d}\zeta_1,\ldots,\mathrm{d}\zeta_r\}$ to a basis $\{\mathrm{d}\zeta_1,\ldots,\mathrm{d}\zeta_r,\mathrm{d}\zeta_{r+1},\ldots,\mathrm{d}\zeta_n\}$ of $\mathscr{X}$. Then, in the coordinates $\{\zeta_1,\ldots,\zeta_n\}$, the system can be decomposed into observable and unobservable subsystems

$$
\begin{aligned}
\zeta_1^\Delta &= f_1\left(\zeta_1,\ldots,\zeta_r,u\right),\\
&\vdots\\
\zeta_r^\Delta &= f_r\left(\zeta_1,\ldots,\zeta_r,u\right),\\
y &= h\left(\zeta_1,\ldots,\zeta_r\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\zeta_{r+1}^\Delta &= f_{r+1}\left(\zeta,u\right),\\
&\vdots\\
\zeta_n^\Delta &= f_n\left(\zeta,u\right),
\end{aligned}
$$

respectively.

Example 4.5. (Continuation of Example 4.3). Integrating the observable space $\mathscr{O}_\infty$ of the system, we get the set of the observable state variables $\zeta_1 = x_1 - x_2$ and $\zeta_2 = x_3$. Next we complete this set to a basis $\{\zeta_1, \zeta_2, \zeta_3\}$ of $\mathbb{R}^3$, taking, for example, $\zeta_3 = x_1$. In these coordinates the system equations read as

$$\begin{aligned} \zeta_1^\Delta &= u_1, \\ \zeta_2^\Delta &= u_2 \cos^2 \zeta_2, \\ \zeta_3^\Delta &= u_1 \tan \zeta_2, \\ y_1 &= \zeta_1, \\ y_2 &= \zeta_2, \end{aligned}$$

where the first two equations (together with the output equations) define the observable subsystem. The state ζ_3 is unobservable.

5. CONCLUSIONS

Although the theory of continuous- and discrete-time dynamical systems as presented in the literature is different, the analysis of time scales is nowadays recognized as a right tool to unify the seemingly separate fields of discrete dynamical systems (i.e. difference equations) and continuous dynamical systems (i.e. differential equations). In the paper we studied the observability of multi-input multi-output control systems on a homogeneous time scale, which allows us to unify continuous- and discrete-time theories, presenting both of them simultaneously under the same language. The presented approach covers the continuous- and discrete-time cases in such a manner that those are special cases of the formalism. Since the delta derivative (used in our paper to describe the dynamical systems) coincides with the time derivative for the continuous-time case, the results available in the literature can be obtained from our results as a special case, namely the case in which the time scale is a set of real numbers. On the other hand, our formalism includes the description of a discrete-time system based on the difference operator description (delta-domain approach), for which the results shown in the paper are new, since previous results have been obtained for discrete-time systems considered on the basis of the shift-operator formalism. Therefore, in our paper the discrete-time systems are described in terms of the difference operator, unlike in the majority of papers where the system is described via the shift-operator. To conclude, although the computation of the delta derivative is different in the continuous- and discrete-time cases, the results obtained by means of it are the same for both time domains.

In the paper the notion of the observable space was used to provide the observability condition that can be easily checked. However, note that the definition of the observability was introduced through the observability rank condition, commonly used both in continuous- and discrete-time cases. One of the future goals is to define the observability of the nonlinear system on a homogeneous time scale, using the concept of (in)distinguishable states. Another goal is to find the conditions under which the nonlinear system defined on a homogeneous time scale is transformable into the observer form, which allows construction of an observer with linearizable error dynamics.

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APPENDIX

PROOF OF LEMMA 3.4

In order to prove Lemma 3.4, we need Lemma 5.1 below.

Lemma 5.1. *For the homogeneous time scale $\mathbb{T}$ one has*

$$\frac{\partial h_\nu^{[i+1]}}{\partial x} = \frac{\partial h_\nu^{[i]}}{\partial x}\frac{\partial f(x,u)}{\partial x} + \left(\frac{\partial h_\nu^{[i]}}{\partial x}\right)^{\Delta_f}\left(I_n + \mu\frac{\partial f(x,u)}{\partial x}\right), \quad \nu = 1,\dots p, \quad i = 0,1,\dots, \tag{24}$$

where I_n is the $n\times n$ identity matrix.

Proof. By commutativity of operators d and Δ_f [4],

$$\mathrm{d}\left(h_\nu^{[i+1]}\right) = \left(\mathrm{d}h_\nu^{[i]}\right)^{\Delta_f}. \tag{25}$$

In what follows, we omit in (25) the parts involving the terms $\mathrm{d}u_\upsilon^{[l]}$ in the expressions of total differentials, therefore we have

$$\frac{\partial h_\nu^{[i+1]}}{\partial x}\mathrm{d}x + \cdots = \left(\frac{\partial h_\nu^{[i]}}{\partial x}\mathrm{d}x\right)^{\Delta_f} + \cdots. \tag{26}$$

We compute the delta derivative of the one-form in the right-hand side of (26), using (4). Since $(\mathrm{d}x)^{\Delta_f} = \mathrm{d}f(x,u)$, and again, omitting the parts involving the terms $\mathrm{d}u_\upsilon$, we get

$$\left(\frac{\partial h_\nu^{[i]}}{\partial x}\mathrm{d}x\right)^{\Delta_f} = \left(\frac{\partial h_\nu^{[i]}}{\partial x}\right)^{\Delta_f}\mathrm{d}x + \left(\frac{\partial h_\nu^{[i]}}{\partial x}\right)^{\sigma_f}\frac{\partial f(x,u)}{\partial x}\mathrm{d}x + \cdots.$$

Since the vectors $\mathrm{d}x$, $\mathrm{d}u_\upsilon$,..., $\mathrm{d}u_\upsilon^{[i-1]}$ are independent over the field $\mathscr{K}^*$, by comparing the coefficients of $\mathrm{d}x$ at both sides of equality (26) we get

$$\frac{\partial h_\nu^{[i+1]}}{\partial x} = \left(\frac{\partial h_\nu^{[i]}}{\partial x}\right)^{\Delta_f} + \left(\frac{\partial h_\nu^{[i]}}{\partial x}\right)^{\sigma_f}\frac{\partial f(x,u)}{\partial x}.$$

Finally, applying (i) of Proposition 2.4 to $\left(\frac{\partial h_\nu^{[i]}}{\partial x}\right)^{\sigma_f}$, we obtain (24). □

Now we are ready to prove Lemma 3.4.

Proof. According to the condition of the lemma,

$$\omega_{\nu,i} := \frac{\partial h_\nu^{[i]}}{\partial x}\mathrm{d}x = \sum_{k=0}^{i-1}\alpha_k\frac{\partial h_\nu^{[k]}}{\partial x}\mathrm{d}x. \tag{27}$$

We first prove that the statement of the lemma holds for $j = i+1$, i.e.

$$\omega_{\nu,i+1} = \sum_{k=0}^{i-1}\beta_k\frac{\partial h_\nu^{[k]}}{\partial x}\mathrm{d}x = \sum_{k=0}^{i-1}\beta_k\omega_{\nu,k} \tag{28}$$

for some β_k's. By Lemma 5.1 and (27)

$$\omega_{\nu,i+1} = \sum_{k=0}^{i-1} \left[\alpha_k \frac{\partial h_\nu^{[k]}}{\partial x} \frac{\partial f(x,u)}{\partial x} + \left(\alpha_k \frac{\partial h_\nu^{[k]}}{\partial x} \right)^{\Delta_f} \left(I_n + \mu \frac{\partial f(x,u)}{\partial x} \right) \right] \mathrm{d}x.$$

Using (iii) of Proposition 2.4 for $\left(\alpha_k \frac{\partial h_\nu^{[k]}}{\partial x}\right)^{\Delta_f}$ and then (i) of Proposition 2.4 for α_k, we get

$$\omega_{\nu,i+1} = \sum_{k=0}^{i-1} \left[\frac{\partial h_\nu^{[k]}}{\partial x} \left(\frac{\partial f(x,u)}{\partial x} \alpha_k^{\sigma_f} + \alpha_k^{\Delta_f} \right) + \alpha_k^{\sigma_f} \left(\frac{\partial h_\nu^{[k]}}{\partial x} \right)^{\Delta_f} \left(I_n + \mu \frac{\partial f(x,u)}{\partial x} \right) \right] \mathrm{d}x.$$

By Lemma 5.1

$$\left(\frac{\partial h_\nu^{[k]}}{\partial x} \right)^{\Delta_f} \left(I_n + \mu \frac{\partial f(x,u)}{\partial x} \right) = \frac{\partial h_\nu^{[k+1]}}{\partial x} - \frac{\partial h_\nu^{[k]}}{\partial x} \frac{\partial f(x,u)}{\partial x},$$

yielding

$$\omega_{\nu,i+1} = \sum_{k=0}^{i-1} \alpha_k^{\Delta_f} \frac{\partial h_\nu^{[k]}}{\partial x} \mathrm{d}x + \sum_{k=0}^{i-1} \alpha_k^{\sigma_f} \frac{\partial h_\nu^{[k+1]}}{\partial x} \mathrm{d}x.$$

Changing the summation index of the second sum for $s = k+1$, separating the last addend of the second sum, and applying (27) to it, we obtain

$$\omega_{\nu,i+1} = \sum_{k=0}^{i-1} \left(\alpha_k^{\Delta_f} + \alpha_{i-1}^{\sigma_f} \alpha_k \right) \frac{\partial h_\nu^{[k]}}{\partial x} \mathrm{d}x + \sum_{s=1}^{i-1} \alpha_{s-1}^{\sigma_f} \frac{\partial h_\nu^{[s]}}{\partial x} \mathrm{d}x.$$

Separating the first addend of the first sum yields

$$\omega_{\nu,i+1} = \sum_{k=1}^{i-1} \left(\alpha_k^{\Delta_f} + \alpha_{i-1}^{\sigma_f} \alpha_k + \alpha_{k-1}^{\sigma_f} \right) \frac{\partial h_\nu^{[k]}}{\partial x} \mathrm{d}x + \left(\alpha_0^{\Delta_f} + \alpha_{i-1}^{\sigma_f} \alpha_0 \right) \frac{\partial h_\nu}{\partial x} \mathrm{d}x.$$

Denoting $\beta_0 := \alpha_0^{\Delta_f} + \alpha_{i-1}^{\sigma_f} \alpha_0$ and $\beta_k := \alpha_k^{\Delta_f} + \alpha_{i-1}^{\sigma_f} \alpha_k + \alpha_{k-1}^{\sigma_f}$, we get (28). Similar arguments can be applied to the case $j > i+1$. □

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Homogeensel ajaskaalal defineeritud mittelineaarse juhtimissüsteemi vaadeldav ruum

Vadim Kaparin, Ülle Kotta ja Małgorzata Wyrwas

On uuritud homogeensel ajaskaalal defineeritud mittelineaarse juhtimissüsteemi vaadeldavust, mis tähendab võimalust määrata (leida) süsteemi mittemõõdetav algolek mõõdetavate juhttoimete ja väljundite abil. Vaadeldavuse tingimus on esitatud vaadeldava ruumi mõiste kaudu. Juhul kui süsteem ei ole vaadeldav, aga vaadeldav ruum, mille elementideks on diferentsiaalsed üksvormid, on täielikult integreeruv, on süsteem dekomponeeritav vaadeldavaks ja mittevaadeldavaks alamsüsteemiks.