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Transformation of an irregular wave field along a quartic bottom profile

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Abstract. Random wave transformation in the basin of decreasing depth is studied for the case of a quartic bottom profile. The advantage of this bottom profile is that waves can propagate along it without inner reflection even if the bottom slope is not small. Wave transformation is studied analytically in the framework of shallow-water theory. Its rigorous solution is obtained in the class of random functions. The correlation function and its spectrum (energetic wave spectrum) are calculated. The behaviour of wave spectrum transformation in a basin of decreasing depth is studied in detail. It is demonstrated that the spectrum becomes upshifted while approaching the coast, with its high-frequency asymptotic ω^{-3}.

Key words: waves, coastal zone, wave spectra, shallow water theory, quartic bottom profile.

1. INTRODUCTION

It is known that a decrease in water depth leads to wave amplification and increase in wave amplitude. Usually, these processes are studied numerically taking into account a real bathymetry of the basin, wave refraction, wave diffraction, nonlinearity, and breaking processes. The possibility of using the analytical approach is very limited. However, even very specific analytical solutions are profitable, since they can be used as benchmarks for tests of numerical codes. Analytical methods and some solutions can be found in many works (LeBlond and Mysak, 1978; Mei, 1983; Massel, 1989; Dingemans, 1996; Pelinovsky, 1996; Mei et al., 2005). These methods have been used for benchmarks of long wave runup on a beach with an application to tsunami (Liu et al., 2008). A special class of analytical solutions concerns so-called “nonreflecting” bottom configurations, when the wave propagates over large distances without inner reflection from the bottom slope (Clements and Rogers, 1975; Bluman and Kumei, 1987; Tinti et al., 2001; Choi et al., 2008; Didenkulova et al., 2008; Didenkulova and Pelinovsky, 2009, 2011; Grimshaw et al., 2010). Similar effects have also been recorded in acoustics (Ibragimov and Rudenko, 2004) and in atmosphere and solar physics (Petrukhin et al., 2011, 2012; Cally, 2012). The main advantage of these bottom configurations is that the solution of the Cauchy problem is significantly simplified and can be found explicitly without using any integral transformations like Fourier transformation. Such solutions can be very convenient for analytical analysis of wave propagation in the coastal zone.

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In this paper we study wave transformation along one of these nonreflecting configurations, a quartic bottom profile ($h \sim x^4$) (Didenkulova and Pelinovsky, 2010).

2. TRAVELLING WAVE SOLUTION

The basic equations for long water waves of small amplitude in a basin of variable depth are

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[h(x)u] = 0, \qquad \frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} = 0, \tag{1}$$

where η is the water displacement, u is the depth-averaged water flow velocity, $h(x)$ is an unperturbed water depth, g is gravity acceleration, t is time, and x is the coordinate, which is directed offshore in the problem considered. Formally, shallow-water equations are derived from the Euler equation with the use of small parameter depth/wavelength, which lead to hydrostatic approximation, and should not be applied to steep bottom profiles. However, it was shown by Dingemans (1996), Massel (1996), and Mei et al. (2005) that shallow-water equations work well for the description of the wave field even in these cases.

It is convenient to reduce this system to two variable-coefficient wave equations for the water displacement η and flow velocity u

$$\frac{\partial^2 \eta}{\partial t^2} - g\frac{\partial}{\partial x}\left[h(x)\frac{\partial \eta}{\partial x}\right] = 0, \tag{2}$$

$$\frac{\partial^2 u}{\partial t^2} - g\frac{\partial^2}{\partial x^2}[h(x)u] = 0. \tag{3}$$

Equations (2) and (3) are equivalent, but not interchangeable. In fact, the substitution $w = \partial\eta/\partial x$ allows transformation of Eq. (2) into Eq. (3), but the function $w(x,t)$ describes wave steepness and not water flow, which indeed can be determined by integration of $w(x,t)$ over time; see the second equation in Eqs (1). This is why both Eqs (2) and (3) can be used independently for finding rigorous solutions of Eqs (1). For instance, it has been shown that Eq. (2) has a travelling wave solution along the following bottom profile (Clements and Rogers, 1975; Didenkulova et al., 2009):

$$h(x) = h_0\left(\frac{x}{L}\right)^{4/3}, \tag{4}$$

while Eq. (3) also has a travelling wave solution, but along another bottom profile (Didenkulova and Pelinovsky, 2010):

$$h(x) = h_0\left(\frac{x}{L}\right)^{4}. \tag{5}$$

Here h_0/L characterizes the bottom slope which does not need to be small.

Below we study wave transformation above a quartic bottom profile (5). The solution describing the travelling wave propagating onshore has the following form:

$$\eta(x,t) = F(t+\tau) + T_0\left[\frac{h(x)}{h_0}\right]^{-1/4}\frac{dF(t+\tau)}{dt}, \tag{6}$$

$$u(x,t) = -\left(\frac{h(x)}{h_0}\right)^{-3/4}\frac{dF(t+\tau)}{dt},$$
$$\tau(x) = \int\frac{dx}{\sqrt{gh(x)}} = \frac{T_0\,|x-L|}{x}, \tag{7}$$

where $T_0 = L/\sqrt{gh_0}$, h_0 is the water depth at some reference point, L is the distance from this point to the shore. The function $F(t)$ has an evident physical meaning; it describes the water displacement far offshore, for example, measurements by buoy in the open sea. We point out that the arbitrary function $F(t)$ is differentiated in order to have the water flow bounded, otherwise the wave breaks on. Some examples of regular wave transformation above a quartic bottom profile are given in Didenkulova and Pelinovsky (2010). Here we consider a random wave field, which usually reflects wind wave properties.

3. SPECTRUM OF RANDOM WAVES IN THE COASTAL ZONE

In strict sense, in the framework of the mathematical theory of random processes, Eqs (6), (7) cannot be used for the description of a random wave field since random functions are not differentiated or integrated explicitly. That is why below we use another standard physical approach, where the random wave field is represented by a superposition of spectral components with deterministic amplitudes A_i and random phases φ_i (Massel, 1996)

$$\eta(t) = \sum_{i=1}^{N} A_i\cos(\omega_i t + \varphi_i). \tag{8}$$

Then all statistical characteristics can be found as ensemble average. Usually, phases φ_i are assumed to be uniformly distributed in the interval $[0, 2\pi]$. Spectral amplitudes are determined by the energetic frequency spectrum of wind waves $S(\omega)$

$$A_i = \sqrt{S(\omega_i)\Delta\omega_i}, \tag{9}$$

where $\Delta\omega_i$ are intervals of spectrum discretization, which do not need to be equidistant. The number of harmonics in Eq. (8) should be large enough in order to "cover" the energetic spectrum in a wide range.

It is known that the rigorous definition of an energetic spectrum is a Fourier spectrum of an autocorrelation function (Rytov et al., 1989; Gurbatov et al., 1991). Using Eq. (6), the autocorrelation function can be calculated as follows:

$$K(t_1, t_2; x) = <\eta(t_1+\tau)\eta(t_2+\tau)> = <F(t_1+\tau)F(t_2+\tau)>$$
$$+T_0\left(\frac{h}{h_0}\right)^{-1/4}\left[\left\langle F(t_1+\tau)\frac{dF(t_2+\tau)}{dt_2}\right\rangle + \left\langle \frac{dF(t_1+\tau)}{dt_1}F(t_2+\tau)\right\rangle\right]$$
$$+T_0^2\left(\frac{h}{h_0}\right)^{-1/2}\left\langle \frac{dF(t_1+\tau)}{dt_1}\frac{dF(t_2+\tau)}{dt_2}\right\rangle, \qquad (10)$$

where <> is the ensemble averaging. We assume that the random wave process in the open sea is stationary, whose autocorrelation function is a function of time difference ξ only:

$$K(\xi) = <F(t_1+\tau)F(t_1+\zeta+\tau)>. \qquad (11)$$

It can be shown (see, for example, Rytov et al., 1989; Gurbatov et al., 1991) that for a stationary process

$$\left[\left\langle F(t_1+\tau)\frac{dF(t_2+\tau)}{dt_2}\right\rangle + \left\langle \frac{dF(t_1+\tau)}{dt_1}F(t_2+\tau)\right\rangle\right] = 0. \qquad (12)$$

Hence, Eq. (10) is simplified to

$$K(\zeta; x) = K(\zeta) - T_0^2\left(\frac{h(x)}{h_0}\right)^{-1/2}\frac{d^2K(\zeta)}{d\zeta^2}. \qquad (13)$$

It follows from Eq. (13) that the random process is stationary at any point in space, but the magnitude of the autocorrelation function depends on coordinate through the local depth $h(x)$. The autocorrelation function is a deterministic function of a single parameter. Therefore, using a Fourier transformation, which is well determined, the required formula for the energetic spectrum in the coastal zone can be derived:

$$S(\omega, h) = S(\omega)\left[1 + \omega^2 T_0^2\left(\frac{h}{h_0}\right)^{-1/2}\right]. \qquad (14)$$

Equation (14) is valid for any approximation of a wave spectrum in the open sea $S(\omega)$. In the next section we apply it to the case of the Pierson–Moskowitz spectrum (Massel, 1996).

4. SPECTRUM EVOLUTION ABOVE A QUARTIC BOTTOM PROFILE

Several approximations of the wind wave spectrum can be found, for instance, in Massel (1996). Here we use the Pierson–Moskowitz spectrum

$$S(\omega) = 8.1\times10^{-3}\frac{g^2}{\omega^5}\exp\left[-0.74\left(\frac{W\omega}{g}\right)^{-4}\right], \qquad (15)$$

where W is the wind speed that is assumed constant. This approximation is used for a fully developed wind wave field in open sea.

For a detailed description of the wind wave spectrum in the coastal zone it is convenient to introduce dimensionless variables

$$\Omega = \frac{W\omega}{g}, \qquad T(h) = \frac{gT_0}{W}\left(\frac{h}{h_0}\right)^{-1/4},$$
$$S(\omega, h) = 8.1\times10^{-3}\frac{W^5}{g^3}E(\Omega, h). \qquad (16)$$

Hence, we can find a dimensionless wave spectrum from Eq. (15)

$$E(\Omega, h) = \frac{1+T^2\Omega^2}{\Omega^5}\exp\left(-\frac{0.74}{\Omega^4}\right). \qquad (17)$$

As the dimensionless spectrum is determined by a single parameter T only, its asymptotics are easy to analyse. In the low-frequency range, the spectrum does not depend on the water depth and coincides with the Pierson–Moskowitz spectrum

$$E(\Omega \to 0, h) = \frac{1}{\Omega^5}\exp\left(-\frac{0.74}{\Omega^4}\right), \qquad (18)$$

and for high frequencies its asymptotic follows:

$$E(\Omega \to \infty, h) = T^2(h)\frac{1}{\Omega^3}. \qquad (19)$$

So, spectral amplitudes of short waves in the coastal zone increase proportionally to $h^{-1/2}$ with their high-frequency asymptotic ω^{-3}, which coincides with the saturated tail of frequency spectra in shallow water proposed by Thornton (1977).

This difference between spectrum transformation in low and high frequencies can be explained by the characteristic distance in terms of wavelengths, which waves require in order to change their properties. This implies that long waves almost do not change and short waves are amplified significantly at the same absolute distance.

The shape of the spectrum (17) is plotted in Fig. 1 for different values of T, which depend on the water depth.

The spectrum becomes wider and slightly shifts to higher frequencies with an increase in T, which corresponds to a decrease in water depth. The amplitude of the spectral peak also grows with an increase in T. The frequency of the spectrum maximum, Ω_0, changes from 0.877 to 0.997 depending on T as shown in Fig. 2.

The standard deviation of the spectrum (17) can be found from

$$\sigma^2(T) = \int_0^\infty d\Omega \frac{1+T^2\Omega^2}{\Omega^5} \exp\left(-\frac{0.74}{\Omega^4}\right), \tag{20}$$

and demonstrated in Fig. 3. For small values of T (deep water) σ remains essentially constant while in more shallow regions (large values of T) $\sigma \sim T \sim h^{-1/4}$.

It is important that the value of 2σ defines the significant wave height for wind waves in oceanography (average of 1/3 of the highest waves) and the value of 4σ determines the threshold for identification of rogue or freak waves. All waves, whose height is larger than 4σ, are rogue (Slunyaev et al., 2011). Such waves occasionally appear at the sea surface, leading to human losses, ship accidents, and damage to offshore and coastal structures (Nikolkina and Didenkulova, 2011, 2012). It has recently been shown that rogue waves appear more frequently in coastal areas (Nikolkina and Didenkulova, 2011; Slunyaev et al., 2011). The same rogue wave philosophy is also used in optics, acoustics, gas dynamics, solar physics, etc. (Akhmediev and Pelinovsky, 2010).

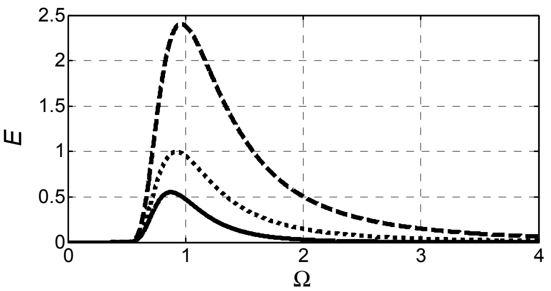

Fig. 1. The spectrum of Eq. (17) for $T = 0$ (solid line), $T = 1$ (dotted line), and $T = 2$ (dashed line).

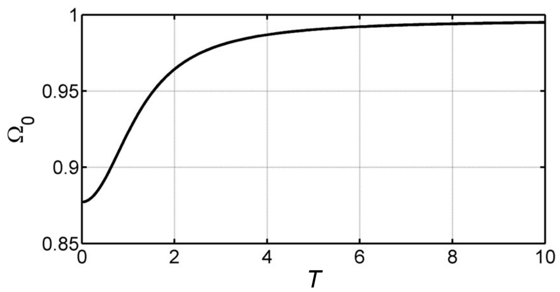

Fig. 2. Frequency of the spectrum maximum as a function of T.

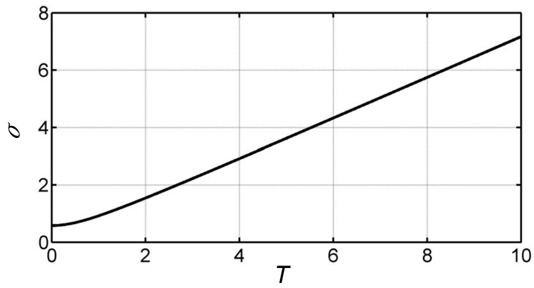

Fig. 3. Standard deviation as a function of T.

It should be noted that, due to the shoaling effect, the wave amplitude increases significantly in the nearshore zone and, therefore, the linear theory approximation no longer works. In shallow water, applicability of the linear theory is judged by the ratio of wave amplitude over depth, which should be small, while in deep water the role of such a small parameter is represented by wave steepness. Nonlinear effects at the non-reflecting beach profile have been recently studied by Didenkulova and Pelinovsky (2012).

5. CONCLUSION

The transformation of irregular waves in the basin of decreasing depth is studied analytically in the framework of shallow-water theory for the case of a quartic bottom profile. Such a profile allows wave propagation over large distances without inner reflection from the bottom even if the bottom slope is not small.

The correlation function and its spectrum (energetic wave spectrum) are calculated. The transformation of the wave spectrum along a quartic bottom profile is studied by the example of the Pierson–Moskowitz spectrum. It is shown that at the low-frequency range, the spectrum does not depend on the water depth and coincides with the Pierson–Moskowitz spectrum, while at the high-frequency range, its spectral amplitudes increase proportionally to $h^{-1/2}$ and a spectrum asymptotic follows ω^{-3}, which coincides with an asymptotic of the shallow-water frequency spectra proposed by Thornton (1977). So, the spectrum in the coastal zone transforms differently in low- and high-frequency ranges. A simple explanation for this is that waves experience changes at the characteristic length depending on the wavelength. Therefore, long waves may almost not change and short waves may be amplified significantly at the same absolute distance.

The standard deviation of the spectrum, which is related to the value of the significant wave height in oceanography, is also calculated. It is shown that in deep water it is almost constant and changes insignificantly, while in shallower regions it is proportional to $h^{-1/4}$.

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Juhusliku lainevälja spektri teisenemine neljandat järku rannaprofiilil

Ira Didenkulova ja Efim Pelinovsky

Analüütiliste meetoditega ja madala vee teooria raames on analüüsitud ühemõõtmelises juhuslikus laineväljas toimuvaid protsesse olukorras, kus lained levivad otse ranna poole mööda rannaprofiili, kus vee sügavus $h \sim x^4$ muutub võrdeliselt kaugusega rannast neljandas astmes. Sellistel profiilidel levivate lainete puhul ei toimu peegeldumist isegi suhteliselt suure põhjakalde puhul. On leitud ülesande täpne lahend juhuslike funktsioonide klassis algselt Piersoni-Moskowitzi spektriga lainesüsteemi jaoks. On näidatud, et lainete levimisel mööda sellist profiili energiaspekter teiseneb, lühemad lained võimenduvad ja spektri lühilaineline osa läheneb funktsioonile ω^{-3}, kus ω on lainete ringsagedus.