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MECHANICS

Deformation waves in microstructured solids and dimensionless parameters

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Abstract. On the basis of the Mindlin-type micromorphic theory for wave motion in microstructured solids the 1D governing equations and corresponding dispersion relations are derived. The leading physical dimensionless parameters are established and their importance for describing dispersion effects is discussed. The general discussion reveals the role of both geometrical and physical dimensionless parameters in mechanics of microstructured materials.

Key words: continuum mechanics, microstructures, stress waves, dimensionless parameters.

1. INTRODUCTION

The mathematical models which describe wave motion in microstructured solids, as a rule, involve many physical parameters. In the celebrated paper by Mindlin [1] the number of parameters in the most general 3D case involves as many as 1764 physical constants. Clearly, this number is too large to be determined by physical experiments and thus further studies have focused on specifying the leading parameters, including Mindlin [1] himself. The crucial point is to establish the leading physical effects and then derive the corresponding governing equation using various theories [2–4]. In addition, various models are compared between themselves [5–9]. It is clear that besides the dispersion analysis, attention must be paid to specifying the main parameters. Following Barenblatt [10], the scaling of physical parameters against the basic notions will give a better insight into the character of processes.

In this paper we very briefly describe the Mindlin-type model of microstructured solids, focusing on final governing equations of motion. Then the dimensionless parameters are brought up and their significance is discussed. The discussion is centred around the importance of geometrical and physical dimensionless parameters which characterize microstructured solids.

2. MATHEMATICAL MODELS

We use the Mindlin-type micromorphic model [1] modified by Engelbrecht et al. [3]. It has been shown that such a model can be linked to models derived also by other assumptions [3,11], therefore providing an excellent basis for deriving the governing equations for microstructured solids.

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Leaving aside the details, the starting point in modelling is the free energy function W, which is taken for the 1D case in the form

$$W = \frac{A}{2}u_x^2 + \frac{B}{2}\varphi^2 + \frac{C}{2}\varphi_x^2 + D\varphi u_x + \frac{N}{6}u_x^3 + \frac{M}{6}\varphi_x^3, \tag{1}$$

where A, B, C, D, N, M are material parameters, u is the macrodisplacement, and φ is the microdeformation. Here and further, the subscripts denote differentiation with respect to the indicated variable.

Together with the kinetic energy

$$K = \frac{1}{2}\rho u_t^2 + \frac{1}{2}I\varphi_t^2, \tag{2}$$

where ρ is macrodensity and I is microinertia, the governing equations of motion can be derived by making use of the Euler–Lagrange equation for the Lagrangian $\mathscr{L} = K - W$. The corresponding equations of motion are then (cf. [3])

$$\rho u_{tt} = A u_{xx} + D\varphi_x + N u_x u_{xx}, \tag{3a}$$

$$I\varphi_{tt} = C\varphi_{xx} - B\varphi - D u_x + M\varphi_x\varphi_{xx}. \tag{3b}$$

After introducing dimensionless variables $X = x/L$, $T = tc_0/L$, $U = u/U_0$, the scaling parameter $\delta = l^2/L^2$ (L and U_0 can be the wavelength and the amplitude of the initial excitation, respectively; $c_0^2 = A/\rho$ and l is the characteristic scale of the microstructure), and making use of the slaving principle [12], the following hierarchical model is obtained from Eqs (3):

$$U_{TT} - \left(1 - \frac{c_A^2}{c_0^2}\right)U_{XX} - \frac{\mu}{2}\left(U_X^2\right)_X = \delta\left(\beta U_{TT} - \gamma U_{XX} - \delta^{1/2}\frac{\lambda}{2}U_{XX}^2\right)_{XX}. \tag{4}$$

Here the following notations are used:

$$c_A^2 = \frac{D^2}{\rho B}, \quad \beta = \frac{ID^2}{\rho l^2 B^2}, \quad \gamma = \frac{CD^2}{AB^2 l^2}, \quad \mu = \frac{NU_0}{AL}, \quad \lambda = \frac{D^3 M U_0}{AB^3 l^3 L}. \tag{5}$$

The notion of wave hierarchy is introduced by Whitham [13] for description of scaling the different wave operators. It means that in a wave hierarchy scale parameters indicate the dominance of certain wave operators. This is exactly the case of Eq. (4) where parameter δ has this role: if δ is small, the wave operator on the left-hand side of Eq. (4) is dominant and if δ is large, the wave operator on the right-hand side of Eq. (4) is dominant.

If coefficients are determined in terms of speeds only, Eq. (4) yields

$$U_{TT} - \left(1 - \frac{c_A^2}{c_0^2}\right)U_{XX} - \frac{1}{2}k_N\left(U_X^2\right)_X = \frac{c_A^2}{c_B^2}\left(U_{TT} - \frac{c_1^2}{c_0^2}U_{XX}\right)_{XX} + \frac{1}{2}k_M\left(U_{XX}^2\right)_{XX}. \tag{6}$$

Here the following notations are used:

$$c_B^2 = \frac{BL^2}{I}, \quad c_1^2 = \frac{C}{I}, \tag{7}$$

and k_N, k_M are the parameters expressing the strengths of physical nonlinearities on macro- and microscale, respectively.

The linear approximation of Eq. (6)

$$U_{TT} - \left(1 - \frac{c_A^2}{c_0^2}\right)U_{XX} = \frac{c_A^2}{c_B^2}\left(U_{TT} - \frac{c_1^2}{c_0^2}U_{XX}\right)_{XX} \tag{8}$$

demonstrates clearly the hierarchical nature of the process. Here the coefficient c_A^2/c_B^2 has the role of the scaling parameter.

If we return to initial variables, the full system (3) with $k_N = k_M = 0$ (i.e., the linear case) and the hierarchical approximation (6) can be written as

$$u_{tt} - (c_0^2 - c_A^2)u_{xx} = -p^2(u_{tt} - c_0^2 u_{xx})_{tt} + p^2 c_1^2 (u_{tt} - c_0^2 u_{xx})_{xx} \tag{9}$$

and

$$u_{tt} - (c_0^2 - c_A^2)u_{xx} = p^2 c_A^2 (u_{tt} - c_1^2 u_{xx})_{xx}, \tag{10}$$

respectively [14]. Here the time parameter p is defined as $p^2 = \frac{I}{B}$.

In order to derive the dispersion relations, the solution

$$u(x,t) = \hat{u}\exp[i(kx - \omega t)] \tag{11}$$

is assumed. Then Eqs (9) and (10) yield the following dispersion relations:

$$\omega^2 = (c_0^2 - c_A^2)k^2 + p^2(\omega^2 - c_0^2 k^2)(\omega^2 - c_1^2 k^2), \tag{12}$$

$$\omega^2 = (c_0^2 - c_A^2)k^2 - p^2 c_A^2(\omega^2 - c_1^2 k^2)k^2, \tag{13}$$

respectively.

Detailed analysis of nonlinear [15,16] and linear [14,17] cases gives insight into the wave profile distortions in these complicated models. Distortions of wave profiles can be related to the differences in phase (c_{ph}) and group (c_{gr}) speeds.

3. PHYSICAL DIMENSIONLESS PARAMETERS FOR MICROSTRUCTURED SOLIDS

Following the models in Section 2, we specify the parameters [3,17]

$$\gamma_A^2 = \frac{c_A^2}{c_0^2} = \frac{D^2}{AB}, \tag{14}$$

$$\gamma_1^2 = \frac{c_1^2}{c_0^2} = \frac{\rho C}{AI} = \frac{C^*}{AI^*}, \tag{15}$$

$$\gamma_{AB}^2 = \frac{c_A^2}{c_B^2} = \frac{D^2 I}{\rho B^2 L^2} = \delta \frac{I^* D^2}{B^2}, \tag{16}$$

and [16]

$$\Gamma = 1 - \gamma_A^2 - \gamma_1^2, \tag{17}$$

where it is assumed that $I = \rho l^2 I^*$ and $C = l^2 C^*$, and I^* is dimensionless and C^* has the dimension of stress. Introduction of I^* and C^* is needed for the proper scaling in order to derive hierarchical equation (4) [3].

Parameter Γ is crucial to distinction between the dispersion type following the acoustic dispersion branch. If $\Gamma > 0$, the dispersion is normal ($c_{\text{gr}} < c_{\text{ph}}$) (see Fig. 1a) and if $\Gamma < 0$, the dispersion is anomalous ($c_{\text{gr}} > c_{\text{ph}}$) (see Fig. 1b). If $\Gamma = 0$, we have the dispersionless case. The optical dispersion branch always describes the case $c_{\text{gr}} < c_{\text{ph}}$.

The influence of normal and anomalous dispersion on the character of solution is demonstrated by solving system (3) in its linear form ($N = M = 0$) under a sinusoidal boundary condition for the material initially at rest (see [17] for details). For calculations the Laplace transform is used together with the numerical evaluation of the inverse transform [17]. The results are depicted in Fig. 2a (normal dispersion) and Fig. 2b (anomalous dispersion), respectively. Although the boundary condition has constant frequency, the fastest part of the signal is made of many frequencies and the signal disperses according to the dispersion

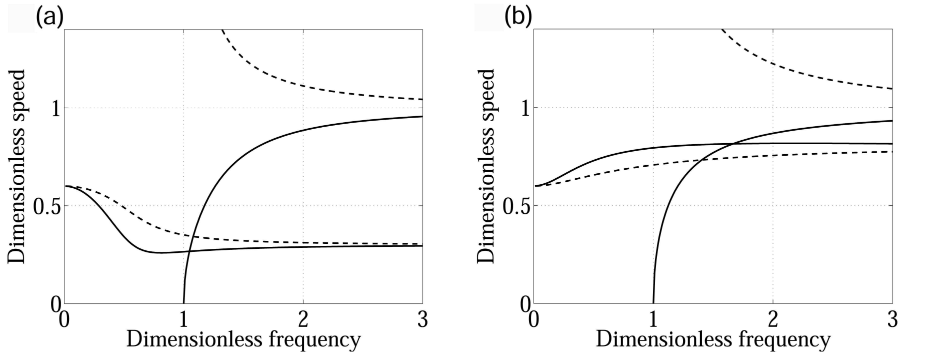

Fig. 1. Group (solid line) and phase (dashed line) speed curves for (a) $\gamma_A = 0.8$ and $\gamma_1 = 0.3$ and (b) for $\gamma_A = 0.8$ and $\gamma_1 = 0.8$.

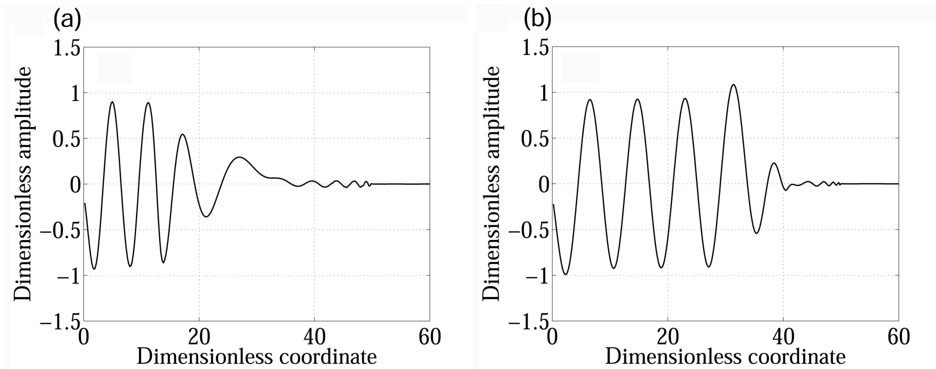

Fig. 2. Wave profiles at 50 time steps for (a) $\gamma_A = 0.8$ and $\gamma_1 = 0.3$ and (b) for $\gamma_A = 0.8$ and $\gamma_1 = 0.8$. The dimensionless frequency at the boundary is equal to 0.5.

type. The low-amplitude oscillations in Fig. 2 reflect the influence of the optical dispersion branch, which is a direct consequence of the inclusion of the microstructure [18].

Parameter γ_A is directly related to coupling effects and influences the speed of the wave. Such an effect has also been demonstrated by numerical calculations [3]. Parameter γ_1 is actually the ratio of speeds in micro- and macrostructures while parameter γ_{AB} is related to inertia of the microelement and coupling effects. In addition, parameters γ_A and γ_1 define the dimensionless speed of long ($(1-\gamma_A^2)^{1/2}$) and short waves (γ_1), respectively. The greater the parameter γ_A, the smaller the speed of long waves and the greater the parameter γ_1, the greater the speed of short waves. Returning to initial coefficients in the free energy function (1), it is obvious that the smaller the value of A, B, and I, the greater the value of γ_A and γ_1 and the greater the value of C, D, and ρ, the greater the value of γ_A and γ_1.

Similarly to Fig. 2, in Fig. 3 the influence of parameter γ_A on the wave motion is demonstrated by solving system (3) in its linear form under a sinusoidal boundary condition for the material initially at rest. In order to consider a simple dispersionless case, γ_1 should be altered as well. It can be seen that in case of the smaller value of parameter γ_A (Fig. 3a), the high-amplitude part of the wave profile travels faster than in case of the higher value of parameter γ_A (Fig. 3b). The low-amplitude part travelling at the unit speed reflects again the influence of the optical dispersion branch.

Parameters γ_A and γ_1 together can be used for estimating the differences between full equation (9) and its approximation (10). For example, it is possible to estimate the regions in the $\gamma_A - \gamma_1$ plane where the dispersion curves for the acoustic branch of Eqs (9) and (10) differ with the accuracy of 5% or 10% at a given frequency or wave number [14]. These differences will be translated into differences in wave profiles, while the speed of the main pulse/signal is equal in both models [17,18].

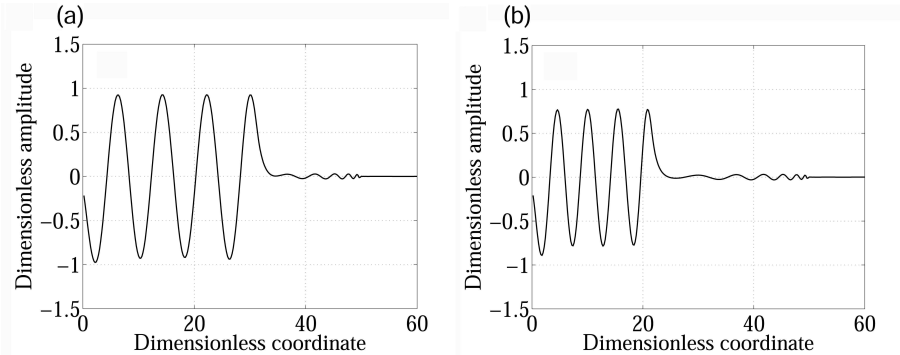

Fig. 3. Wave profiles at 50 time steps for (a) $\gamma_A = 0.77$ and $\gamma_1 = 0.63$ and (b) for $\gamma_A = 0.95$ and $\gamma_1 = 0.31$. Dimensionless frequency is equal to 0.5.

4. THE IMPORTANCE OF DIMENSIONLESS PARAMETERS

In general terms it is clear that both geometry and physical properties of microstructure(s) influence the wave motion in microstructured solids. Here we present a brief summary on parameters and their importance.

4.1. Geometrical parameters

The most important effect of the geometry is the emphasized ratio of the scale length l of the microstructure to the wavelength of the excitation L. We have used $\delta = l^2/L^2$ for describing the hierarchical model (4). As shown above, if δ is small, the waves are governed by the properties of the macrostructure, if, however δ is large, the waves 'feel' more microstructure (see for example [3]).

A similar parameter is of importance for granular materials where the scale length is the particle diameter [19]. However, the variety of microstructures in technical as well as in biological materials needs more elaborated analysis for understanding the influence of grains, cells, particles, pores, etc. on macrobehaviour. It is proposed to use stereology and 3D microscopy for the quantitative analysis of microstructures [20]. In this case, for example, the notion of contiguity is introduced, which characterizes the fraction of the spatial area shared with other elements (grains) of the microstructure.

For porous structures like in electrodes from a lithium-ion battery, the notion of tortuosity is introduced [21], which relates the minimum distance within a pore to the shortest distance between pixels. The crucial problem is how these geometrical parameters reflect the physics (and physical properties) of solids.

For wood, which is a highly cellular material, the geometry of cells can be linked to the density of the cellular structures ρ and the density of the solid cell wall ρ_s [22]. For Voronoi honeycombs (foams) a dimensionless parameter characterizes the regularity of honeycombs [23], for ceramics dimensionless parameters characterize the ratio of the solid–solid and solid–void surface area (surface area ratio) and the ratio of mean grain and mean void intercepts (intercept ratio) [24].

4.2. Physical parameters

Besides technological materials, snow can also be characterized in terms of a microstructured medium [25]. In this case the microstructural index I_s is introduced as

$$I_s = S_V/N_{BV}L_3^2, \tag{18}$$

where S_V is the mean grain surface area per unit volume, N_{BV} is the mean number of bonds per unit volume, and L_3 represents the grain character. The other physical parameters are shown to be dependent on microstructural index I_s which actually combines physical and geometrical parameters.

For wave motion attention should be paid to dispersion effects. Maugin [26] has introduced a parameter with dimensions in order to characterize waves in elastic crystals for which the governing equations are derived from lattice dynamics. For waves in martensitic-austenitic alloys, where shear effects are important, this parameter in his notations is

$$\alpha = \beta - \frac{1}{2}c_T^2, \tag{19}$$

where c_T is the leading velocity for shear motion and parameter β is related to shear-deformation coupling effects which are based on the special form of the potential [26]. Parameter α is shown to govern the conditions of soliton existence. Clearly, α can be represented in a dimensionless form

$$\alpha' = \frac{\alpha}{c_T^2} = \frac{\beta}{c_T^2} - \frac{1}{2}. \tag{20}$$

A model for gradient elastic solids as another form of Mindlin theory [1] has been derived by Papargyri-Beskou et al. [9]. In order to compare their results with ours, we rewrite Eq. (14) from [9] in its 1D form:

$$u_{tt} - c_0^2 u_{xx} = h^2 u_{xxtt} - g^2 c_0^2 u_{xxxx}, \tag{21}$$

where c_0 is the conventional sound speed and $g^2 > 0$, $h^2 > 0$ characterize microstructural effects – stiffness of the microstructure and its inertia, respectively, and have dimensions of length square. Equation (21) is clearly similar to the hierarchical approximation (10) in our studies. It is shown in [9] that if $h > g$, dispersion is normal and if $h < g$, dispersion is anomalous. This statement can be reformulated in the dimensionless way: if $h/g > 1$, dispersion is normal and if $h/g < 1$, dispersion is anomalous. Note that g^2 is related to potential energy and h^2 to kinetic energy. Such a physical background is characteristic also of granular media. For this case Giovine and Oliveri [27] have constructed an evolution equation (a one-wave equation) as a hierarchical Korteweg–de Vries (KdV)-type model for longitudinal waves. They showed that the sign of the higher-order operator, which is reflected also in the sign of the higher-order dispersive term, is related to the ratio of contributions from potential and kinetic energies. The same effect for our model (10) is evident from an evolution equation derived by M. Randrüüt (see [14]), where the sign before the dispersion term in the governing KdV-type equation characterizes the convexity or concavity of the dispersion curve.

5. CONCLUSIONS

We demonstrated that physical parameters γ_A, γ_1, and Γ characterize dispersion effects in microstructured solids. Besides elasticity like in models derived from lattice dynamics [26], these parameters involve also inertial effects of the embedded microstructure. Actually these parameters are related to speeds of waves including also the coupling between macro- and microstructure. The influence of γ_A and γ_1 can be seen from the analysis of initial and boundary value problems, while Γ governs the character of dispersion.

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Deformatsioonilained mikrostruktuuriga materjalides ja dimensioonita parameetrid

Jüri Engelbrecht, Tanel Peets, Kert Tamm ja Andrus Salupere

Baseerudes Mindlini tüüpi mikromorfsel teoorial, mis kirjeldab lainelevi mikrostruktuuriga tahkistes, on tuletatud liikumisvõrrandid 1D juhul ja neile vastavad dispersiooniseosed. Liikumisvõrrandites on leitud olulised füüsikalised dimensioonita parameetrid ja kirjeldatud nende mõju dispersiooniefektidele. Nimetatud parameetrid väljendavad makro- ja mikrostruktuuri seoste mõju ning arvestavad mikrostruktuuri elastsuse ja inertsiga. On selgitatud geomeetriliste ja füüsikaliste dimensioonita parameetrite osatähtsust mikrostruktuuriga materjalide mehaanika kontekstis.