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MECHANICS

Some comments on the theory of short fibre reinforced materials

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Abstract. The orientation of fibres in short fibre reinforced materials is essential for the properties of the composite. It is state of the art to use an orientation number to estimate how many fibres are aligned in the stress direction. This, however, is a very crude approach, as the orientation number is defined by use of the average of the projected length of the fibres. Therefore, the orientation number is not a material property – it depends also on the projection direction. Additionally, a simple number cannot be used to describe anisotropic properties. We introduce a tensorial approach, which is objective and consists of real material properties.

Key words: microstructured solids, constitutive theory, composites, short fibres, alignment tensors.

1. INTRODUCTION

Fibre reinforced materials become increasingly important for constructions of all kinds, from airplanes to buildings. The materials used to form the composite also cover a broad range, including among others glass, carbon or steel for the fibres and metal, plastic or concrete for the matrix. In this paper steel fibre reinforced concrete is chosen as one example. Steel fibre reinforced concrete is a microstructured material, which consists of several homogeneous media (filler, cement, steel fibres), i.e., this material has a basic matrix which includes embedded short metal fibres (see Fig. 1).

Fig. 1. Steel fibre reinforced concrete after a fracture test.

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Today steel fibre reinforced concrete is perhaps one of the most important new possibilities of developing the use of concrete in load-bearing structures. Despite the fact that the properties of this material have not been thoroughly explored, it is already widely used in the construction industry. Fibre reinforced concrete is widely used in floors resting on soil and to a lesser extent in floor slabs, walls, and foundations, but the request to use it also in various load-bearing structural components is high. The investigation of fibre reinforced concrete as building material is very important for the construction area. Its use increases the effectiveness of the designer's and constructor's work, improving also the quality of the structures. The use of fibre reinforced concrete in construction allows gaining considerable achievement in work performance and of course in time.

Failure mechanisms and corresponding calculation methods are still under development when analysing fibre reinforced concrete. As already known, when considering concrete at microscale, the structure of this material is heterogeneous. But when conventional reinforcement is added to the tensioned structure, in macroscale it becomes an orthotropic material. Due to the presence of short fibres in fibre reinforced concrete, one has to consider this material at meso- and microscales. Thus another variable, characterizing the length of the inner structure of the material and associated with the orientation of fibres, has to be added to the governing equations. Determining the location of fibres in the matrix is one of the most important starting points for further development of design rules. In connection with different structural scales, pure theory of reinforced concrete structures cannot be applied to fibre reinforced concrete. Though, there is very much in common between reinforced and fibre reinforced concrete, which can be used in the integration process of concrete and fibre reinforced concrete.

The fibre orientation is important, because the fibres aligned in the direction of major stresses carry more load than fibres perpendicular to this direction.

The major aspects influencing fibre orientations include the wall-effects introduced by the form-work, geometry of the casting element, the way concrete is poured into the mould, the effect of vibration or, in case of self-compacting concrete, the flow of fresh concrete.

2. THE ORIENTATION NUMBER

The average fibre orientation in a certain direction is generally considered through the so-called orientation number [1–3]. This parameter is frequently applied in experimental investigations to quantify the influence of one of the aforementioned aspects on fibre orientation by isolating it from the others. For instance, the effect of the casting direction has been quantified by considering elements poured in different positions while keeping all the remaining aspects constant.

The orientation number is defined as

$$\eta = \frac{1}{N}\sum_{i=1}^{N}\cos\theta_i, \tag{1}$$

where N is the number of fibres in the sample and θ is the angle between the fibre and the surface normal (usually parallel to the principal forces). The orientation number corresponds to the average projected length of the fibres in a plane/cross-section onto the normal of the cross-section, divided by the fibre length.

The in-plane angle is not taken into account. Because of this and as the orientation of fibres is 3D, the orientation number does not allow making any statement about the orientation with respect to an axis perpendicular to the cross-section normal.

3. THE ORIENTATION PROFILE

Using an orientation profile [3], the contribution of fibres at different angles to the load direction is introduced into a constitutive function. The orientation profile is based on the mean orientation angle and respective standard deviation (variance) of measured fibre orientations and an *assumed* statistical density distribution function, e.g. normal distribution. From this one can calculate the amount of fibres at any given angle. The in-plane angle is also in this case not taken into account. For this reason and since the orientation of fibres is 3D, also the orientation profile does not allow making any statement about the orientation with respect to an axis perpendicular to the cross-section normal, or any other axis.

An example of the constitutive function is given in [3]:

$$\sigma_{\mathrm{SF}}(w) = \frac{P_{N,7.5^\circ}+P_{N,22.5^\circ}+P_{N,37.5^\circ}+P_{N,52.5^\circ}+P_{N,67.5^\circ}+P_{N,82.5^\circ}}{A_{\mathrm{sec}}}, \tag{2}$$

where $P_{N,\theta}$ is the pullout response of N fibres in a cross-section A_{sec} under the inclination angle $\theta \in \{7.5^\circ, 22.5^\circ, 37.5^\circ, 52.5^\circ, 67.5^\circ, 82.5^\circ\}$. The partition in 15°-angles seems to be arbitrary.

4. CRITICISM ON THE ORIENTATION NUMBER AND ORIENTATION PROFILE

Neither the orientation number nor the orientation profile are material properties, as they also depend on the projection direction. The orientation number describes only an average orientation, but many different orientation distributions can produce the same average. Additionally, only the angle with respect to the projection direction is taken into account, while the in-plane angle within the cross-section is neglected. Therefore, also the orientation profile does not allow making conclusions about differently oriented cross-sections.

It is known from other examples, like electric permittivity, that tensorial quantities are required to describe the behaviour of anisotropic materials. Neither the orientation number nor the orientation profile have tensorial character.

A formulation based on an orientation distribution function and its alignment tensors would be more elegant. It would take into account the 3D character of the fibre orientation and could represent a contiguous distribution.

5. THE ORIENTATION DISTRIBUTION FUNCTION

The orientation distribution function describes the orientation of fibres, using spherical polar coordinates. As the length of the fibres is constant, a unit-sphere (and therefore only the two angular coordinates) is used (see Fig. 2). Each fibre produces two points (opposing each other) on the sphere. The density of the points is normalized, so that the integral over the whole unit sphere is 1; this density is defined as the orientation distribution function. This concept is well known from mesoscopic continuum physics for liquid crystals (where it was first introduced) and micro-cracks [4], [5: sec. 5.4.5], [6], [7].

The advantage of using the orientation distribution function instead of the orientation number and orientation profile is that both angles are taken into account (with their corresponding profiles). However, it is difficult to obtain the precise orientation distribution function and also to include the orientation distribution function into constitutive equations. For this purpose the alignment tensor is a better alternative.

The orientation distribution function is a field, i.e., it is position-dependent. Naturally, the orientation distribution function will be different in the bulk, compared to that close to the wall or corners, where boundary-effects will dominate.

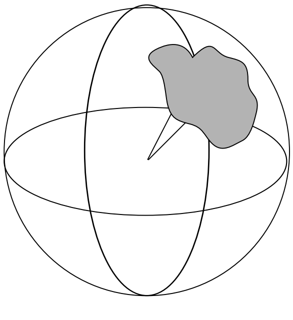

Fig. 2. Orientation distribution function. The symmetric part corresponding to $-\mathbf{n}$ is not shown.

6. THE ALIGNMENT TENSORS

A distribution function can be expanded into a series by using the multipole-expansion. The orientation distribution function is a function $f : S^2 \to \mathbb{R}$. If the orientation distribution function is at least an L^2-function, the expansion is given by [5: sec. 5.4.5]

$$f(\mathbf{n}) = \frac{1}{4\pi}\left(\oint_{S^2} f(\mathbf{n})\, d^2 n + \sum_{l=1}^{\infty} \frac{(2l+1)!!}{l!} a_{\mu_1 \dots \mu_l}\, \overline{n_{\mu_1} \cdots n_{\mu_l}} \right), \tag{3}$$

$$a_{\mu_1 \dots \mu_l} = \oint_{S^2} f(\mathbf{n})\, \overline{n_{\mu_1} \cdots n_{\mu_l}}\, d^2 n. \tag{4}$$

Here, $l!$ is the factorial $l! = l \cdot (l-1) \cdots 2 \cdot 1$ and $(2l+1)!! = (2l+1) \cdot (2l-1) \cdots 3 \cdot 1$ denotes the "factorial with double steps", and $\overline{n_{\mu_1} \cdots n_{\mu_l}}$ are the components of a tensor obtained by the l-fold symmetric tensorial product of the vector $\mathbf{n} \in S^2$, from which the reducible parts have been removed. The tensor basis is given by $\{ \overline{n_{\mu_1} \cdots n_{\mu_l}} \}_{l \in \mathbf{N}_0}$.

Vice versa, the distribution function can be calculated if all alignment tensors are known. If only some moments are known, an approximation to the distribution density function can be obtained. The accuracy of this approximation depends on the number of known moments.

As the orientation distribution function is position-dependent, also the alignment tensors are fields (position-dependent). The use of alignment tensors to describe the orientation order has been introduced in mesoscopic continuum physics for liquid crystals and microcracks [5: sec. 5.4.5].

If the orientation distribution function is symmetric with respect to $\mathbf{n}$, i.e. $f(-\mathbf{n}) = f(\mathbf{n})$, the alignment tensors of odd order vanish. As the fibres usually have a head-tail symmetry, the orientation distribution function will be symmetric.

Depending on the accuracy with which the orientation distribution function shall be represented, it is sufficient to use only the first (couple) of the alignment tensors, although for the *exact* representation, in general an infinite number of alignment tensors is necessary. Already the use of only the second-order alignment tensor is a huge improvement over the current approaches.

If the orientation distribution function is anisotropic, one can define a *macroscopic director* $\mathbf{d}$, which is a certain average orientation direction of the fibres. The macroscopic director is a unit vector that points into the direction of the eigenvector of the according-to-amount-largest eigenvalue of the second-order alignment tensor. In the case of a uniaxial distribution it is the symmetry axis of the distribution. In addition, it is useful to introduce an (orientational) order parameter $S \in [-\frac{1}{2}, 1]$ to account for the amount of anisotropy, where $S = 1$

corresponds to total alignment and $S = 0$ to isotropy; $S = -\frac{1}{2}$ describes a rare situation, when all fibres are aligned in a plane perpendicular to the eigenvector of the first eigenvalue. In the following, it is assumed that the eigenvalues are sorted according to the amount $|\lambda_1| \geq |\lambda_2| \geq |\lambda_2|$. The order parameter and eigenvalues λ_i are related as follows:

$$\frac{2}{3}S = \lambda_1, \quad (5)$$

$$-\frac{1}{3}S - b_S = \lambda_2, \quad (6)$$

$$-\frac{1}{3}S + b_S = \lambda_3, \quad (7)$$

where $b_S = \mathrm{sign}(S)b$ and the biaxiality $b \in \left[0, \frac{1}{3}|S|\right]$ of the distribution [8].

Although the order parameter is connected to the eigenvalues of the order parameter tensor (second-order alignment tensor), one can calculate it as the average of the second Legendre polynomial $P_2(\cos\alpha)$:

$$S := \langle P_2(\cos\alpha) \rangle, \quad (8)$$

$$S = \left\langle \frac{3}{2}\cos^2\alpha - \frac{1}{2} \right\rangle, \quad (9)$$

where α is the angle between the fibre axis (microscopic director) and the macroscopic director. Legendre polynomials are a special case of spherical harmonics.

In contrast to the orientation number, the order parameter is defined with respect to the average fibre orientation, but each one takes only one angle into account. The alignment tensors, however, take both angles into account and one can calculate the number of fibres aligned in any direction by projecting the alignment tensors.

7. CONSTITUTIVE LAWS USING ALIGNMENT TENSORS

Making use of the previously introduced alignment tensors, one can formulate a general constitutive function for steel fibre reinforced concrete (SFRC):

$$T^{ij}_{\mathrm{SFRC}} = T^{ij}_{\mathrm{SFRC}}(T^{kl}_{\mathrm{concrete}}, \mu, a^{kl}, a^{klmn}, T^{kl}_{\mathrm{fibres}}), \quad (10)$$

where $T^{kl}_{\mathrm{concrete}}$ is the stress tensor of concrete without fibres, and T^{kj}_{fibres} is defined to be the stress tensor of an isotropic fibre orientation distribution and independent of the fibre density. It includes the different response of fibres of different inclination angles. The anisotropy is taken into account by the alignment tensors a^{kl} and a^{klmn} and the amount of fibres and spatial inhomogeneities by the volume fraction μ of the fibres. All of these quantities are field functions. Furthermore, properties of the fibres, such as $\frac{d}{l}$, the ratio of diameter to length, and pull-out resistance, enter either T^{kl}_{fibres} or κ, where κ is a scalar material parameter that is to be determined by experiments.

Two hypothetic examples for constitutive mappings are

$$T^{ij}_{\mathrm{SFRC}} = (1-\mu)T^{ij}_{\mathrm{concrete}} + \mu a^{ik} T^{kl}_{\mathrm{fibres}} a^{lj} \quad (11)$$

and

$$T^{ij}_{\mathrm{SFRC}} = (1-\mu)T^{ij}_{\mathrm{concrete}} + \mu \kappa a^{ij}. \quad (12)$$

Fourth-order alignment tensors could be included in these equations by tracing over two indices.

A thermodynamic consistent constitutive theory for (long) fibre reinforced elastic materials has been developed in [9]. However, this work covers only materials where the length of the fibres is the same as the length of the body that is reinforced. A constitutive theory for short fibre reinforced materials is still a subject of research.

8. CONCLUSIONS

In this paper different methods of describing the orientation of fibres in short fibre reinforced composite were discussed. It was clearly shown that the two commonly used methods for steel fibre reinforced concrete, the orientation number and the orientation profile, are rather approximative and thus insufficient. The newly proposed alignment tensors and corresponding (orientational) order parameter and macroscopic director overcome these limitations. The order parameter has the advantage over the orientation number in being related to the expectation value of the second Legendre polynomial, a special case of spherical harmonics, which are well known in electrodynamics and quantum mechanics. Due to its relation to the alignment tensor, the order parameter is objective and a material property. Especially by use of higher-order alignment tensors, it is possible to describe the orientation distribution of the fibres accurately in three dimensions. In addition, the alignment tensors can be easily used to formulate constitutive equations. These constitutive mappings are tensor equations and contain only objective quantities, especially they are independent of any pre-chosen projection direction. They can be used to calculate the stresses in any direction. Furthermore, all quantities are fields, i.e., they will vary along the material. Although steel fibre reinforced concrete has been chosen as an example, the methods are applicable to all kinds of short fibre reinforced composites.

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Kommentaare lühikeste kiududega armeeritud materjalide teooriale

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Lühikiududega armeeritud komposiitmaterjalide omadused on tundlikud kiudude orientatsiooni suhtes. Pingete mõjusuunas orienteeritud kiudude arvu hindamiseks kasutatakse orientatsiooni numbrit. See viib väga jämedale hinnangule, sest orientatsiooni number on defineeritud, kasutades pingete mõjusuunale projekteeritud kiudude pikkuste keskmist väärtust. Seega ei ole orientatsiooni number materjali omadus – see sõltub samuti projektsiooni suunast. Lisaks sellele ei piisa materjali anisotroopsete omaduste kirjeldamiseks ühest numbrist. Artiklis on esitatud tensorarvutusel põhinev lähenemisviis vaadeldud materjalide objektiivsete omaduste kirjeldamiseks.