Proceedings of the Estonian Academy of Sciences,
2011, **60**, 3, 165–173
doi: 10.3176/proc.2011.3.04
Available online at www.eap.ee/proceedings

MATHEMATICS

Para-hyperhermitian structures on tangent bundles

Dedicated to the memory of Professor Stere Ianuş (1939–2010)

Gabriel Eduard Vîlcu

University of Bucharest, Faculty of Mathematics and Computer Science, Research Center in Geometry, Topology and Algebra, Academiei Str. 14, Sector 1, Bucharest 70109, Romania; and Petroleum-Gas University of Ploieşti, Department of Mathematics and Computer Science, Bulevardul Bucureşti 39, Ploieşti 100680, Romania; gvilcu@upg-ploiesti.ro

Received 22 June 2010, revised 22 February 2011, accepted 23 February 2011

Abstract. In this paper we construct a family of almost para-hyperhermitian structures on the tangent bundle of an almost para-hermitian manifold and study its integrability. Also, the necessary and sufficient conditions are provided for these structures to become para-hyper-Kähler.

Key words: para-hyperhermitian structure, tangent bundle, paracomplex space form.

1. INTRODUCTION

The almost para-hypercomplex structures, also named almost quaternionic structures of second kind, were introduced by Libermann in 1954 under the latter name [20] as a triple of endomorphisms of the tangent bundle $\{J_1, J_2, J_3\}$, in which J_1 is almost complex and J_2, J_3 are almost product structures satisfying relations of anti-commutation. An almost para-hyperhermitian structure on a manifold consists of an almost para-hypercomplex structure and a compatible semi-Riemannian metric necessarily of neutral signature. If all three structures involved in the definition of an almost para-hyperhermitian structure are parallel with respect to the Levi-Civita connection of the compatible metric, one arrives at the concept of para-hyper-Kähler structure, which is also referred to in the literature as neutral hyper-Kähler or hypersymplectic structure [4,7,10,13].

The quaternionic structures of second kind are of great interest in theoretical physics, because they arise in a natural way both in string theory and integrable systems [2,6,9,14,23] and, consequently, to find new classes of manifolds endowed with structures of this kind is an interesting topic. Kamada [19] proves that any primary Kodaira surface admits para-hyper-Kähler structures, whose compatible metrics can be chosen to be flat or nonflat. On the other hand, an integrable para-hyperhermitian structure has been constructed in [17] on Kodaira–Thurston properly elliptic surfaces and also on the Inoe surfaces modelled on Sol_1^4. In higher dimensions, para-hyperhermitian structures on a class of compact quotients of 2-step nilpotent Lie groups can be found in [12]. A procedure to construct para-hyperhermitian structures on R^{4n} with complete and not necessarily flat associated metrics is given in [1]. Also, some examples of integrable almost para-hyperhermitian structures which admit compatible linear connections with totally skew symmetric torsion are given in [18]. Recently, in [15], a natural para-hyperhermitian structure was constructed on the tangent bundle of an almost para-hermitian manifold and on the circle bundle over a manifold with a mixed 3-structure. The main purpose of this paper is to generalize this construction to obtain an entire class of such structures; we also investigate its integrability and obtain the necessary and sufficient conditions for these structures to become para-hyper-Kähler.

2. PRELIMINARIES

An almost product structure on a smooth manifold M is a tensor field P of type (1,1) on M, $P \neq \pm Id$, such that

$$P^2 = Id,$$

where Id is the identity tensor field of type (1,1) on M.

An almost para-hermitian structure on a differentiable manifold M is a pair (P,g), where P is an almost product structure on M and g is a semi-Riemannian metric on M satisfying

$$g(PX,PY) = -g(X,Y)$$

for all vector fields X,Y on M.

In this case, (M,P,g) is said to be an almost para-hermitian manifold. It is easy to see that the dimension of M is even. Moreover, if $\nabla P = 0$, then (M,P,g) is said to be a para-Kähler manifold.

An almost complex structure on a smooth manifold M is a tensor field J of type (1,1) on M such that

$$J^2 = -Id.$$

An almost para-hypercomplex structure on a smooth manifold M is a triple $H = (J_\alpha)_{\alpha=\overline{1,3}}$, where J_1 is an almost complex structure on M and J_2, J_3 are almost product structures on M, satisfying

$$J_2 J_1 = -J_1 J_2 = J_3.$$

In this case (M,H) is said to be an almost para-hypercomplex manifold.

A semi-Riemannian metric g on (M,H) is said to be compatible or adapted to the almost para-hypercomplex structure $H = (J_\alpha)_{\alpha=\overline{1,3}}$ if it satisfies

$$g(J_\alpha X, J_\alpha Y) = \varepsilon_\alpha g(X,Y), \forall \alpha = \overline{1,3} \tag{1}$$

for all vector fields X,Y on M, where $\varepsilon_1 = 1, \varepsilon_2 = \varepsilon_3 = -1$. Moreover, the pair (g,H) is called an almost para-hyperhermitian structure on M and the triple (M,g,H) is said to be an almost para-hyperhermitian manifold. It is clear that any almost para-hyperhermitian manifold is of dimension $4m$, $m \geq 1$, and any adapted metric is necessarily of neutral signature $(2m,2m)$. If $\{J_1,J_2,J_3\}$ are parallel in respect to the Levi-Civita connection of g, then the manifold is called para-hyper-Kähler.

An almost para-hypercomplex manifold (M,H) is called a para-hypercomplex manifold if each J_α, $\alpha = 1,2,3$, is integrable, that is, if the corresponding Nijenhuis tensors

$$N_\alpha(X,Y) = [J_\alpha X, J_\alpha Y] - J_\alpha[X, J_\alpha Y] - J_\alpha[J_\alpha X, Y] - \varepsilon_\alpha[X,Y], \tag{2}$$

$\alpha = 1,2,3$, vanish for all vector fields X,Y on M. In this case H is said to be a para-hypercomplex structure on M. Moreover, if g is a semi-Riemannian metric adapted to the para-hypercomplex structure H, then the pair (g,H) is said to be a para-hyperhermitian structure on M and (M,g,H) is called a para-hyperhermitian manifold. We note that the existence of para-hyperhermitian structures on compact complex surfaces was recently investigated in [8].

Remark 2.1. Let (M,P,g) be an almost para-hermitian manifold and TM be the tangent bundle, endowed with the Sasakian metric

$$G(X,Y) = (g(KX,KY) + g(\pi_* X, \pi_* Y)) \circ \pi$$

for all vector fields X,Y on TM, where π is the natural projection of TM onto M and K is the connection map (see [11]).

We remark that for each $u \in T_xM$, $x \in M$, we have a direct sum decomposition

$$T_uTM = T_u^hTM \oplus T_u^vTM,$$

where $T_u^hTM = KerK|_{T_uTM}$ is called the horizontal subspace of T_uTM and $T_u^vTM = Ker\pi_*|_{T_uTM}$ is called the vertical subspace of T_uTM. Moreover, the elements of T_u^hTM are called horizontal vectors at u and the elements of T_u^vTM are said to be vertical vectors at u.

We can see that if $u, X \in T_xM$ and X_u^h (resp. X_u^v) denotes the horizontal lift (resp. vertical lift) of X to T_uTM then

$$\pi_*X_u^h = X,\ \pi_*X_u^v = 0,\ KX_u^h = 0,\ KX_u^v = X.$$

Remark 2.2. If A is a vector field along π (i.e. a map $A : TM \to TM$ such that $\pi \circ A = \pi$) and A^h (resp. A^v) denotes the horizontal lift (resp. vertical lift) of A (that is: $A^h : u \mapsto A_u^h = A(u)_u^h$ and $A^v : u \mapsto A_u^v = A(u)_u^v$), then any horizontal (vertical) vector field X on TM can be written as $X = A^h$ ($X = A^v$) for a unique vector A along π. If A, B are vector fields along π, then, by generalizing the well-known Dombrowski's lemma [11], Ii and Morikawa [16] showed that the brackets of the horizontal and vertical lifts are given by

$$[A^h, B^h]_u = (\nabla_{A^h}B)_u^h - (\nabla_{B^h}A)_u^h - (R(A(u), B(u))u)_u^v, \tag{3}$$

$$[A^h, B^v] = (\nabla_{A^h}B)^v - (\nabla_{B^v}A)^h, \tag{4}$$

$$[A^v, B^v] = (\nabla_{A^v}B)^v - (\nabla_{B^v}A)^v, \tag{5}$$

where the covariant derivative of a vector field C along π in the direction of $\xi \in T_uTM$, $u \in TM$, is defined as the tangent vector to M at $x = \pi(u)$ given by

$$\nabla_\xi C = (K \circ dC)(\xi).$$

We can also remark that every tensor field T of type (1,1) on M is a vector field along π. Moreover, we have

$$\nabla_{A^v}T = T \circ A, \tag{6}$$

and, if T is parallel,

$$\nabla_{A^h}T = 0. \tag{7}$$

We note that the identity map $id : u \mapsto u$ on TM is a parallel tensor field of type (1,1) on M. Moreover, if $||\cdot||^2$ is the function $u \mapsto ||u||^2 = g(u,u)$ on TM, then we have

$$A^h||\cdot||^2 = 0,\ A^v||\cdot||^2 = 2g(A, id). \tag{8}$$

Remark 2.3. If (M, P, g) is an almost para-hermitian manifold, then we can define three tensor fields J_1, J_2, J_3 on TM by the equalities:

$$\begin{cases} J_1X^h = X^v \\ J_1X^v = -X^h \end{cases},$$

$$\begin{cases} J_2X^h = (PX)^v \\ J_2X^v = (PX)^h \end{cases},$$

$$\begin{cases} J_3X^h = (PX)^h \\ J_3X^v = -(PX)^v \end{cases}.$$

It is easy to see that J_1 is an almost complex structure and J_2, J_3 are almost product structures. We also have the following result (see [15]).

Theorem 2.4. *Let (M,P,g) be an almost para-hermitian manifold. Then the triple $H=(J_\alpha)_{\alpha=\overline{1,3}}$ is an almost para-hypercomplex structure on TM which is para-hyperhermitian with respect to the Sasakian metric G. Moreover, H is integrable if and only if (M,P) is a flat para-Kähler manifold.*

In the next section, following the same techniques as in [3,21,22,24–26], we deform the almost para-hyperhermitian structure given above in order to obtain an entire family of structures of this kind on the tangent bundle of an almost para-hermitian manifold.

3. A FAMILY OF ALMOST PARA-HYPERHERMITIAN STRUCTURE ON THE TANGENT BUNDLE OF A PARA-HERMITIAN MANIFOLD

Lemma 3.1. *Let (M,P,g) be an almost para-hermitian manifold and let J_1 be a tensor field of type $(1,1)$ on TM, defined by*

$$\begin{cases} J_1X_u^h = a(t)X_u^v + b(t)g(X,u)u^v + c(t)g(X,Pu)(Pu)^v \\ J_1X_u^v = m(t)X_u^h + n(t)g(X,u)u^h + p(t)g(X,Pu)(Pu)^h \end{cases} \tag{9}$$

for all vectors $X\in T_{\pi(u)}M$, $u\in T_xM$, $x\in M$, where $t=\|u\|^2$ and a,b,c,m,n,p are differentiable real functions. Then J_1 defines an almost complex structure if and only if

$$am+1=0,\ an(a+tb)-b=0,\ ap(a-tc)-c=0\ . \tag{10}$$

Proof. The conditions follow from the property $J_1^2=-Id$. □

Lemma 3.2. *Let (M,P,g) be an almost para-hermitian manifold and let J_2 be a tensor field of type $(1,1)$ on TM, defined by*

$$\begin{cases} J_2X_u^h = a(t)(PX)_u^v - b(t)g(X,Pu)u^v - c(t)g(X,u)(Pu)^v \\ J_2X_u^v = q(t)(PX)_u^h - r(t)g(X,Pu)u^h - s(t)g(X,u)(Pu)^h \end{cases} \tag{11}$$

for all vectors $X\in T_{\pi(u)}M$, $u\in T_xM$, $x\in M$, where $t=\|u\|^2$ and a,b,c,q,r,s are differentiable real functions. Then J_2 defines an almost product structure if and only if

$$aq-1=0,\ ar(a-tc)-c=0,\ as(a+tb)-b=0. \tag{12}$$

Proof. The conditions follow from the property $J_2^2=Id$. □

Proposition 3.3. *Let (M,P,g) be an almost para-hermitian manifold. Then there exists an infinite class of almost para-hypercomplex structures on TM.*

Proof. We define a tensor field J_3 of type (1,1) on TM by $J_3=J_2J_1$, where J_1,J_2 are given by (9) and (11), such that (10) and (12) are satisfied. We can easily see now that $H=(J_\alpha)_{\alpha=\overline{1,3}}$ is an almost para-hypercomplex structure on TM. □

Proposition 3.4. *Let (M,P,g) be an almost para-hermitian manifold, let $H=(J_\alpha)_{\alpha=\overline{1,3}}$ be the almost para-hypercomplex structure on TM given above and let $\widetilde{G}$ be a semi-Riemannian metric on TM defined by*

$$\begin{cases} \widetilde{G}(X^h,Y^h) = \alpha(t)g(X,Y)+\beta(t)g(X,u)g(Y,u)+\gamma(t)g(X,Pu)g(Y,Pu) \\ \widetilde{G}(X^v,Y^v) = \delta(t)g(X,Y)+\varepsilon(t)g(X,u)g(Y,u)+\theta(t)g(X,Pu)g(Y,Pu) \\ \widetilde{G}(X^h,Y^v) = 0 \end{cases}$$

for all vectors $X, Y \in T_{\pi(u)}M$, $u \in T_xM$, $x \in M$, *where* $t = \|u\|^2$ *and* $\alpha, \beta, \gamma, \delta, \varepsilon, \theta$ *are smooth real functions such that* $\alpha, \alpha + t\beta, \alpha - t\gamma$ *or* $\delta, \delta + t\varepsilon, \delta - t\theta$ *are nowhere null. Then* $\widetilde{G}$ *is adapted to the almost para-hypercomplex structure* $H = (J_\alpha)_{\alpha=\overline{1,3}}$ *if and only if*

$$\begin{cases} \beta + \gamma = 0 \\ \delta a^2 - \alpha = 0 \\ a^2\varepsilon(a+tb)^2 + b\alpha(2a+tb) - \beta a^2 = 0 \\ a^2\theta(a-tc)^2 + c\alpha(2a-tc) + \beta a^2 = 0 \end{cases}. \tag{13}$$

Proof. The conditions (13) are obtained by direct computations using the property (1). □

Corollary 3.5. *There exists an infinite class of almost para-hyperhermitian structures on the tangent bundle of an almost para-hermitian manifold.*

4. THE STUDY OF INTEGRABILITY

Let (M, P, g) be a para-Kähler manifold. A plane $\Pi \subset T_pM$, $p \in M$, is called para-holomorphic if it is left invariant by the action of P, that is $P\Pi \subset \Pi$. The para-holomorphic sectional curvature is defined as the restriction of the sectional curvature to para-holomorphic non-degenerate planes. A para-Kähler manifold is said to be a paracomplex space form if its para-holomorphic sectional curvatures are equal to a constant, say k. It is well known that a para-Kähler manifold (M, P, g) is a paracomplex space form, denoted $M(k)$, if and only if its curvature tensor is

$$\begin{aligned} R(X,Y)Z \;=\; & \frac{k}{4}\{g(Y,Z)X - g(X,Z)Y + g(Y,PZ)PX \\ & -g(X,PZ)PY - 2g(X,PY)PZ\} \end{aligned} \tag{14}$$

for all vector fields X, Y, Z on M.

From the above section we deduce that the tangent bundle of a paracomplex space form $M(k)$ can be endowed with a class of almost para-hypercomplex structure $H = (J_\alpha)_{\alpha=\overline{1,3}}$ given by

$$\begin{cases} J_1X_u^h = aX_u^v + bg(X,u)u^v + cg(X,Pu)(Pu)^v \\ J_1X_u^v = -\frac{1}{a}X_u^h + \frac{b}{a(a+tb)}g(X,u)u^h + \frac{c}{a(a-tc)}g(X,Pu)(Pu)^h \end{cases}, \tag{15}$$

$$\begin{cases} J_2X_u^h = a(PX)_u^v - bg(X,Pu)u^v - cg(X,u)(Pu)^v \\ J_2X_u^v = \frac{1}{a}(PX)_u^h - \frac{c}{a(a-tc)}g(X,Pu)u^h - \frac{b}{a(a+tb)}g(X,u)(Pu)^h \end{cases}, \tag{16}$$

$$\begin{cases} J_3X_u^h = (PX)_u^h \\ J_3X_u^v = -(PX)_u^v + \frac{b+c}{a-tc}g(X,Pu)u^v + \frac{b+c}{a+tb}g(X,u)(Pu)^v, \end{cases} \tag{17}$$

where a, b, c are differentiable real functions such that $\frac{1}{a}, \frac{b}{a(a+tb)}, \frac{c}{a(a-tc)}$ are well defined and also differentiable real functions.

Theorem 4.1. *Let* $M(k)$ *be a paracomplex space form. Then the almost para-hypercomplex structure* $H = (J_\alpha)_{\alpha=\overline{1,3}}$ *given above is integrable if and only if*

$$4b(a - 2ta') - 8aa' + k = 0,\ 4ac + k = 0. \tag{18}$$

Proof. First of all we remark that if two of the structures J_1, J_2, J_3 are integrable, then the third structure is also integrable because the corresponding Nijenhuis tensors are related by

$$\begin{aligned} 2N_\alpha(X,Y) &= N_\beta(J_\gamma X, J_\gamma Y) + N_\gamma(J_\beta X, J_\beta Y) - J_\beta N_\gamma(J_\beta X, Y) \\ &\quad - J_\beta N_\gamma(X, J_\beta Y) - J_\gamma N_\beta(J_\gamma X, Y) - J_\gamma N_\beta(X, J_\gamma Y) \\ &\quad + \varepsilon_\gamma N_\beta(X,Y) + \varepsilon_\beta N_\gamma(X,Y) \end{aligned}$$

for any even permutation (α, β, γ) of (1,2,3), where $\varepsilon_1 = 1, \varepsilon_2 = \varepsilon_3 = -1$.

Secondly, it is well known that an almost complex structure on a manifold is integrable if and only if the distribution of the complex tangent vector fields of type (1,0), denoted by $\mathscr{X}^{(1,0)}$, is involutive, i.e. it satisfies $[\mathscr{X}^{(1,0)}, \mathscr{X}^{(1,0)}] \subset \mathscr{X}^{(1,0)}$. Now, using (2), (3)–(8), and (14), we obtain for any vector field A, B along π, satisfying $g(A, id) = g(B, id) = g(A, P \circ id) = g(B, P \circ id) = 0$ on TM

$$[A^h - iaA^v, B^h - iaB^v]_u = (\nabla_{A^h} B - \nabla_{B^h} A)^h_u - ia(\nabla_{A^h} B - \nabla_{B^h} A)^v_u,$$

$$\begin{aligned} [A^h - iaA^v, (id)^h - i(a+tb)(id)^v]_u = &- [(\nabla_{(id)^h} A)^h_u - ia(\nabla_{(id)^h} A)^v_u] \\ &+ i(a+tb)[(\nabla_{(id)^v} A)^h_u - ia(\nabla_{(id)^v} A)^v_u] \\ &- ia\left[A^h_u - i\frac{\frac{kt}{4} + a(a+tb) - 2a't(a+tb)}{a} A^v_u\right], \end{aligned}$$

$$\begin{aligned} [A^h - iaA^v, (P \circ id)^h - i(a-tc)(P \circ id)^v]_u = &- [(\nabla_{(P\circ id)^h} A)^h_u - ia(\nabla_{(P\circ id)^h} A)^v_u] \\ &+ i(a-tc)[(\nabla_{(P\circ id)^v} A)^h_u - ia(\nabla_{(P\circ id)^v} A)^v_u] \\ &- ia\left[(PA)^h_u - i\frac{-\frac{kt}{4} + a(a-tc)}{a} (PA)^v_u\right], \end{aligned}$$

$$\begin{aligned} &[(id)^h - i(a+tb)(id)^v, (P \circ id)^h - i(a-tc)(P \circ id)^v]_u \\ &\qquad = i(a-tc)\left[(P \circ id)^h_u - i\frac{kt + (a+tb)(a-tc) - 2t(a+tb)(a-tc)'}{a-tc}(P \circ id)^v_u\right] \\ &\qquad - i(a+tb)[(\nabla_{(id)^v} P \circ id)^h - i(a-tc)(\nabla_{(id)^v} P \circ id)^v], \end{aligned}$$

$$[(id)^h - i(a+tb)(id)^v, (id)^h - i(a+tb)(id)^v]_u = 0,$$

$$[(P \circ id)^h - i(a-tc)(P \circ id)^v, (P \circ id)^h - i(a-tc)(P \circ id)^v]_u = 0.$$

Consequently, J_1 is integrable if and only if the next three relations are satisfied:

$$\frac{\frac{k}{4}t + a(a+tb) - 2a't(a+tb)}{a} = a, \tag{19}$$

$$\frac{-\frac{k}{4}t + a(a-tc)}{a} = a, \tag{20}$$

$$\frac{kt+(a+tb)(a-tc)-2t(a+tb)(a-tc)'}{a-tc}=a-tc. \tag{21}$$

Thirdly, an almost product structure on a manifold is integrable if and only if the eigendistributions $\mathscr{X}^+$ and $\mathscr{X}^-$ corresponding to the eigenvalues 1 and -1, respectively, are integrable. We similarly obtain that J_2 is integrable if and only if the same three relations hold.

Finally, we obtain the conclusion because the relation (21) is involved by the relations (19) and (20). □

Example 4.2. If M is a flat paracomplex space form, then we set

$$a=A,\ b=0,\ c=0,$$

where A is an arbitrary non-zero real constant, and we can easily see that the conditions (18) are satisfied and $a,b,c,\frac{1}{a},\frac{b}{a(a+tb)},\frac{c}{a(a-tc)}$ are clearly differentiable, being constants. Consequently, the almost para-hypercomplex structure $H=(J_\alpha)_{\alpha=\overline{1,3}}$ given above is integrable.

Example 4.3. If $M(k)$ is a non-flat paracomplex space form, then we set

$$a=\sqrt[4]{k^2(t^2+A^2+1)},$$

$$b=\frac{\sqrt[4]{k^2(t^2+A^2+1)}(4t-sgn\{k\}\sqrt{t^2+A^2+1})}{4(A^2+1)},$$

$$c=\frac{-k}{4\sqrt[4]{k^2(t^2+A^2+1)}},$$

where A is an arbitrary real constant, and we can easily verify that the conditions (18) are satisfied and the functions $a,b,c,\frac{1}{a},\frac{b}{a(a+tb)},\frac{c}{a(a-tc)}$ are differentiable. Consequently, the almost para-hypercomplex structure $H=(J_\alpha)_{\alpha=\overline{1,3}}$ given above is integrable.

Remark 4.4. From Proposition 3.4 we have a class of compatible semi-Riemannian metric on the tangent bundle of a paracomplex space form $M(k)$, given by

$$\begin{cases} \widetilde{G}(X^h,Y^h)=\alpha g(X,Y)+\beta g(X,u)g(Y,u)-\beta g(X,Pu)g(Y,Pu) \\ \widetilde{G}(X^v,Y^v)=\frac{\alpha}{a^2}g(X,Y)+\frac{\beta a^2-b\alpha(2a+tb)}{a^2(a+tb)^2}g(X,u)g(Y,u) \\ \qquad\qquad\quad +\frac{-\beta a^2-c\alpha(2a-tc)}{a^2(a-tc)^2}g(X,Pu)g(Y,Pu) \\ \widetilde{G}(X^h,Y^v)=0 \end{cases}, \tag{22}$$

where α,β are differentiable real functions such that α and $\alpha+t\beta$ are nowhere null.

From Theorem 4.1 we may state now the following result.

Corollary 4.5. *There exists an infinite class of para-hyperhermitian structures on the tangent bundle of a paracomplex space form.*

Theorem 4.6. *Let $M(k)$ be a paracomplex space form. Then the almost para-hyperhermitian structure $(\widetilde{G},H=(J_\alpha)_{\alpha=\overline{1,3}})$ on TM defined by (15), (16), (17), and (22) is para-hyper-Kähler if and only if M is flat and*

$$a=C_1,\ \alpha=C_2,\ b=c=\beta=0, \tag{23}$$

where C_1,C_2 are non-null real constants.

Proof. If M is flat and the relations (23) hold, then it is clear that $(TM, \widetilde{G}, H = (J_\alpha)_{\alpha=\overline{1,3}})$ is a para-hyper-Kähler manifold.

Conversely, if $(TM, \widetilde{G}, H = (J_\alpha)_{\alpha=\overline{1,3}})$ is a para-hyper-Kähler manifold, then each J_α is integrable and the fundamental 2-forms ω_α, given by

$$\omega_\alpha(X,Y) = \widetilde{G}(J_\alpha X, Y),$$

for all vector fields X, Y on TM, are closed for all $\alpha \in \{1,2,3\}$.

For any vector field A and B along π, satisfying $g(A,id) = g(B,id) = g(A,P\circ id) = g(B,P\circ id) = 0$ on TM, using (3)–(8), we obtain

$$\begin{aligned}(d\omega_1)(A^h,B^v,id^v) &= A^h\widetilde{G}(J_1B^v,id^v) - B^v\widetilde{G}(J_1A^h,id^v) \\ &\quad + id^v\widetilde{G}(J_1A^h,B^v) + \widetilde{G}(J_1 id^v,[A^h,B^v]) \\ &\quad - \widetilde{G}(J_1B^v,[A^h,id^v]) + \widetilde{G}(J_1A^h,[B^v,id^v]) \\ &= \left[2\left(\frac{\alpha}{a}\right)' t + \left(\frac{\alpha}{a} - \frac{\alpha + t\beta}{a+tb}\right)\right] g(X,Y) \end{aligned} \tag{24}$$

and

$$\begin{aligned}(d\omega_1)(A^h,B^v,(P\circ id)^v) &= A^h\widetilde{G}(J_1B^v,(P\circ id)^v) - B^v\widetilde{G}(J_1A^h,(P\circ id)^v) \\ &\quad + (P\circ id)^v\widetilde{G}(J_1A^h,B^v) + \widetilde{G}(J_1(P\circ id)^v,[A^h,B^v]) \\ &\quad - \widetilde{G}(J_1B^v,[A^h,(P\circ id)^v]) + \widetilde{G}(J_1A^h,[B^v,(P\circ id)^v]) \\ &= \left(\frac{\alpha}{a} - \frac{\alpha+t\beta}{a-tc}\right) g(X,PY). \end{aligned} \tag{25}$$

But, because ω_1 is closed, from (24) and (25) we obtain

$$\frac{\alpha}{a} = \frac{\alpha + t\beta}{a+tb} = \frac{\alpha+t\beta}{a-tc} = C, \tag{26}$$

where C is a real constant.

On the another hand, J_1, J_2, J_3 being integrable, from Theorem 4.1 we deduce that the functions a, b, c also satisfy the conditions (18). The conclusion follows now easily since $a, b, c, \alpha, \beta, \frac{1}{a}, \frac{b}{a(a+tb)}, \frac{c}{a(a-tc)}$ must be differentiable functions satisfying (18) and (26). □

5. CONCLUSIONS

We constructed an infinite class of almost para-hyperhermitian structures on the tangent bundle of an almost para-hermitian manifold (M,P,g). Moreover, if (M,P,g) is a paracomplex space form, we also obtained necessary and sufficient conditions for the above structures to become para-hyper-Kähler. These results can have important applications both in differential geometry and theoretical physics, since the existence of para-hyper-Kähler structures is of great importance in many geometric and physics problems (see e.g. [5]). A possible extension of this paper is to construct a class of paraquaternionic Kähler structures on the tangent bundle of a paracomplex space form.

ACKNOWLEDGEMENTS

I would like to express my deepest appreciation to Professor Liviu Ornea for carefully reading this paper and offering me helpful suggestions. This work was partially supported by CNCSIS - UEFISCSU, project PNII - IDEI code 8/2008, contract No. 525/2009.

REFERENCES

1. Andrada, A. and Dotti, G. I. Double products and hypersymplectic structures on R^{4n}. *Commun. Math. Phys.*, 2006, **262**(1), 1–16.
2. Barret, J., Gibbons, G. W., Perry, M. J., Pope, C. N., and Ruback, P. Kleinian geometry and the $N = 2$ superstring. *Int. J. Mod. Phys. A*, 1994, **9**, 1457–1494.
3. Bejan, C. L. and Oproiu, V. Tangent bundles of quasi-constant holomorphic sectional curvatures. *Balkan J. Geom. Appl.*, 2006, **11**(1), 11–22.
4. Blair, D. E., Davidov, J., and Muškarov, O. Hyperbolic twistor space. *Rocky Mt. J. Math.*, 2005, **35**(5), 1437–1465.
5. Cortés, V. The special geometry of Euclidian supersymmetry: a survey. *Rev. Unión Mat. Argent.*, 2006, **47**(1), 29–34.
6. Cortés, V., Mayer, C., Mohaupt, T. and Saueressig, F. Special geometry of euclidean supersymmetry II. Hypermultiplets and the *c*-map. *J. High Energy Phys.*, 2005, **6**, 1–25.
7. Dancer, A. and Swann, A. Hypersymplectic manifolds. In *Recent Developments in Pseudo-Riemannian Geometry* (Alekseevsky, D., ed.). ESI Lectures in Mathematics and Physics, 2008, 97–148.
8. Davidov, J., Grantcharov, G., Mushkarov, O., and Yotov, M. Para-hyperhermitian surfaces. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)*, 2009, **52**(100), 281–289.
9. Dunajski, M. Hyper-complex four-manifolds from Tzitzeica equation. *J. Math. Phys.*, 2002, **43**, 651–658.
10. Dunajski, M. and West, S. Anti-self-dual conformal structures in neutral signature. In *Recent Developments in Pseudo-Riemannian Geometry* (Alekseevsky, D., ed.). ESI Lectures in Mathematics and Physics, 2008, 113–148.
11. Dombrowski, P. On the geometry of the tangent bundle. *J. Reine Angew. Math.*, 1962, **210**, 73–88.
12. Fino, A., Pedersen, H., Poon, Y.-S., and Sørensen, M. W. Neutral Calabi-Yau structures on Kodaira manifolds. *Commun. Math. Phys.*, 2004, **248**(2), 255–268.
13. Hitchin, N. Hypersymplectic quotients. *Acta Acad. Sci. Tauriensis*, 1990, **124**, 169–180.
14. Hull, C. M. Actions for (2,1) sigma models and strings. *Nucl. Phys. B*, 1998, **509**, 252–272.
15. Ianuş, S. and Vîlcu, G. E. Some constructions of almost para-hyperhermitian structures on manifolds and tangent bundles. *Int. J. Geom. Methods Mod. Phys.*, 2008, **5**(6), 893–903.
16. Ii, K. and Morikawa, T. Kähler structures on the tangent bundle of Riemannian manifolds of constant positive curvature. *Bull. Yamagata Univ. Nat. Sci.*, 1999, **14**(3), 141–154.
17. Ivanov, S. and Zamkovoy, S. Para-hermitian and para-quaternionic manifolds. *Differ. Geom. Appl.*, 2005, **23**, 205–234.
18. Ivanov, S., Tsanov, V., and Zamkovoy, S. Hyper-para-Hermitian manifolds with torsion. *J. Geom. Phys.*, 2006, **56**(4), 670–690.
19. Kamada, H. Neutral hyper-Kähler structures on primary Kodaira surfaces. *Tsukuba J. Math.*, 1999, **23**, 321–332.
20. Libermann, P. Sur le problème d'équivalence de certaines structures infinitésimales. *Ann. Mat. Purra Appl.*, 1954, **36**, 27–120.
21. Nakashima, Y. and Watanabe, Y. Some constructions of almost Hermitian and quaternion metric structures. *Math. J. Toyama Univ.*, 1990, **13**, 119–138.
22. Olszak, Z. On almost complex structures with Norden metrics on tangent bundles. *Period. Math. Hung.*, 2005, **51**(2), 59–74.
23. Ooguri, H. and Vafa, C. Geometry of $N = 2$ strings. *Nucl. Phys. B*, 1991, **361**, 469–518.
24. Oproiu, V. and Papaghiuc, N. General natural Einstein Kähler structures on tangent bundles. *Differ. Geom. Appl.*, 2009, **27**(3), 384–392.
25. Tahara, M., Vanhecke, L., and Watanabe, Y. New structures on tangent bundles. *Note Mat.*, 1998, **18**(1), 131–141.
26. Tahara, M., Marchiafava, S., and Watanabe, Y. Quaternionic Kähler structures on the tangent bundle of a complex space form. *Rend. Ist. Mat. Univ. Trieste*, 1999, **31**(1–2), 163–175.

Para-hüperhermiitilised struktuurid puutujakihtkondadel

Gabriel Eduard Vîlcu

On konstrueeritud peaaegu para-hüperhermiitiliste struktuuride parv peaaegu para-hermiitilise muutkonna puutujakihtkonnal ja uuritud selle integreeruvust. On leitud tarvilikud ja piisavad tingimused, mille korral need struktuurid on para-hüper-Kähleri struktuurid.