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MATHEMATICS

On infinitesimal holomorphically projective transformations on the tangent bundles with respect to the Sasaki metric

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Abstract. The purpose of the present article is to find solutions to a system of partial differential equations that characterize infinitesimal holomorphically projective transformations on the tangent bundle with the Sasaki metric and an adapted almost complex structure. Moreover, it is proved that if the tangent bundle of a Riemannian manifold admits a non-affine infinitesimal holomorphically projective transformation, then the Riemannian manifold is locally flat.

Key words: almost complex structures, infinitesimal holomorphically projective transformation, Sasaki metric.

1. INTRODUCTION

Let M be an n-dimensional manifold and $T(M)$ its tangent bundle. We denote by $\Im_s^r(M)$ the set of all tensor fields of type (r,s) on M. Similarly, we denote by $\Im_s^r(T(M))$ the corresponding set on $T(M)$.

The problems of determining infinitesimal holomorphically projective transformation on M and $T(M)$ have been considered by several authors. Ishihara [11] has introduced the notion of infinitesimal holomorphically projective transformation, and Tachibana and Ishihara [18] investigated infinitesimal holomorphically projective transformations on the Kahlerian manifolds. Hasegawa and Yamauchi [8] have proved that 1) infinitesimal holomorphically projective transformation is infinitesimal isometry on a compact Kahlerian manifold with non-positive constant scalar curvature and 2) a compact Kahlerian manifold M with constant scalar curvature is holomorphically isometric to a complex projective space with the Fubini–Study metric if M admits a non-isometric infinitesimal holomorphically projective transformation. In [9,10] they also investigated infinitesimal holomorphically projective transformations on $T(M)$ with respect to horizontal and complete lift connections. Recently, Tarakci, Gezer, and Salimov [19] studied similar problems on $T(M)$ with the metric $II+III$. In [14] new results of the holomorphically projective mappings and transformations were presented.

Starting from a Riemannian connection ∇ of g on M, Sasaki [16] has shown how to construct the Riemannian metric ${}^Sg$ on $T(M)$. This metric has been called *diagonal lift metric*, *Sasaki metric* or *the metric $I+III$*. The Sasaki metric has been extensively studied by many authors, including Gudmundsson and Kappos [7], Kowalski [12], Musso and Tricerri [15], and Aso [4]. Kowalski [12] calculated the Levi-Civita connection ${}^S\nabla$ of the Sasaki metric on $T(M)$ and its Riemannian curvature tensor ${}^SR$. With this in hand, Kowalski [12], Aso [4] and Musso and Tricerri [15] derived interesting connections between the geometric properties of (M,g) and $(T(M),{}^Sg)$.

In this paper, we shall use the method of adapted frames to determine infinitesimal holomorphically projective transformations on $T(M)$ with the Sasaki metric. We also note that here everything will be always discussed in the C^{∞}-category, and manifolds will be assumed to be connected and dimension $n>1$.

2. PRELIMINARIES

2.1. Basic formulas on tangent bundles

Let $T(M)$ be a tangent bundle of M, and π the natural projection $\pi : T(M) \to M$. Let the manifold M be covered by a system of coordinate neighbourhoods (U, x^i), where $(x^i), i = 1, ..., n$ is a local coordinate system defined in the neighbourhood U. Let (y^i) be the Cartesian coordinates in each tangent space $T_p(M)$ at $P \in M$ with respect to the natural base $\left\{\frac{\partial}{\partial x^i}\right\}$, P being an arbitrary point in U whose coordinates are x^i. Then we can introduce local coordinates (x^i, y^i) on the open set $\pi^{-1}(U) \subset T(M)$. We call them *coordinates induced* on $\pi^{-1}(U)$ from (U, x^i). The projection π is represented by $(x^i, y^i) \to (x^i)$. The indices $I, J, ...$ run from 1 to $2n$, the indices $\bar{i}, \bar{j}, ...$ run from $n+1$ to $2n$. Summation over repeated indices is always implied.

Let $X = X^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field X on M. Then the horizontal lift ${}^H X$ and the vertical lift ${}^V X$ of X are given, with respect to the induced coordinates,

$$ {}^V X = X^i \partial_{\bar{i}}, \left(\partial_{\bar{i}} = \frac{\partial}{\partial x^{\bar{i}}}\right), \tag{1} $$

and

$$ {}^H X = X^i \partial_i - y^j \Gamma^i_{jk} X^k \partial_{\bar{i}}, \tag{2} $$

where Γ^i_{jk} are the coefficients of the Levi-Civita connection ∇.

Explicit expression for the Lie bracket [,] of the tangent bundle $T(M)$ is given by Dombrowski [6]. The bracket products of vertical and horizontal vector fields are given by the formulas

$$ \begin{aligned} \left[{}^H X, {}^H Y\right] &= {}^H[X,Y] - \gamma(R(X,Y)), \\ \left[{}^H X, {}^V Y\right] &= {}^V(\nabla_X Y), \\ \left[{}^V X, {}^V Y\right] &= 0 \end{aligned} $$

for all vector fields X and Y on M, where R is the Riemannian curvature of g defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and $\gamma(R(X,Y))$ is a tensor field of type $(1,0)$ on $T(M)$, which is locally expressed as $\gamma(R(X,Y)) = y^s R_{jks}{}^i X^j Y^k \partial_{\bar{i}}$ with respect to the induced coordinates.

The Sasaki metric ${}^S g$ on the tangent bundle $T(M)$ over a Riemannian manifold (M,g) is defined by three equations

$$ {}^S g({}^V X, {}^V Y) = {}^V(g(X,Y)), \tag{3} $$

$$ {}^S g({}^V X, {}^H Y) = 0, \tag{4} $$

$$ {}^S g({}^H X, {}^H Y) = {}^V(g(X,Y)) $$

for all vector fields $X, Y \in \Im^1_0(M)$. It is obvious that the Sasaki metric ${}^S g$ is contained in the class of natural metrics. (Recall that by a natural metric on tangent bundles we shall mean a metric which satisfies conditions (3) and (4)).

2.2. Expressions in adapted frames

With the Riemannian connection ∇ given on M, we can introduce on each induced coordinate neighbourhood $\pi^{-1}(U)$ of $T(M)$ a frame field which is very useful in our computation. It is called the *adapted frame* on $\pi^{-1}(U)$ and consists of the following $2n$ linearly independent vector fields $\{E_\lambda\} = \{E_i, E_{\bar{i}}\}$ on $\pi^{-1}(U)$:

$$E_i = \partial_i - y^b \Gamma^a_{bi} \partial_{\bar{a}}, \;\; E_{\bar{i}} = \partial_{\bar{i}},$$

where $\{x^h, y^h\}$ is the induced coordinates of $T(M)$. $\{dx^h, \delta y^h\}$ is the dual frame of $\{E_i, E_{\bar{i}}\}$, where $\delta y^h = dy^h + y^b \Gamma^h_{ba} dx^a$. From (1) and (2), the vertical lift ${}^V X$ and the horizontal lift ${}^H X$ of $X \in \Im^1_0(M)$ are defined as follows: ${}^H X = X^a E_a$ and ${}^V X = X^a E_{\bar{a}}$ with respect to the adapted frame. By straightforward calculation, we have the following:

Lemma 1. *The Lie brackets of the adapted frame of $T(M)$ satisfy the following identities*:

$$\begin{aligned} [E_j, E_i] &= y^b R_{ijb}{}^a E_{\bar{a}}, \\ [E_j, E_{\bar{i}}] &= \Gamma^a_{ji} E_{\bar{a}}, \\ [E_{\bar{j}}, E_{\bar{i}}] &= 0, \end{aligned}$$

where $R_{ijb}{}^a$ denote the components of the curvature tensor of M [21].

Lemma 2. *Let $\tilde{V}$ be a vector field on $T(M)$, then*

$$\begin{cases} [\tilde{V}, E_i] = -(E_i \tilde{V}^a) E_a + (\tilde{V}^c y^b R_{icb}{}^a - \tilde{V}^{\bar{b}} \Gamma^a_{bi} - E_i \tilde{V}^{\bar{a}}) E_{\bar{a}}, \\ [\tilde{V}, E_{\bar{i}}] = -(E_{\bar{i}} \tilde{V}^a) E_a + (\tilde{V}^b \Gamma^a_{bi} - E_{\bar{i}} \tilde{V}^{\bar{a}}) E_{\bar{a}}, \end{cases}$$

where $\tilde{V} = \begin{pmatrix} \tilde{V}^h \\ \tilde{V}^{\bar{h}} \end{pmatrix} = \tilde{V}^a E_a + \tilde{V}^{\bar{a}} E_{\bar{a}}$ [9].

If $g = g_{ji} dx^j dx^i$ is the expression of the Riemannian metric g on M, the Sasaki metric ${}^S g$ is expressed in the adapted local frame by [6,16]

$${}^S g = g_{ji} dx^j dx^i + g_{ji} \delta y^j \delta y^i.$$

For the Levi-Civita connection of the Sasaki metric, we have the following:

Lemma 3. *Let ${}^S\nabla$ be a Levi-Civita connection of the Sasaki metric ${}^S g$, then*

$$\begin{aligned} {}^S\nabla_{E_j} E_i &= \Gamma^a_{ji} E_a - \frac{1}{2} y^b R_{jib}{}^a E_{\bar{a}}, \\ {}^S\nabla_{E_j} E_{\bar{i}} &= \frac{1}{2} y^b R_{bij}{}^a E_a + \Gamma^a_{ji} E_{\bar{a}}, \\ {}^S\nabla_{E_{\bar{j}}} E_i &= \frac{1}{2} y^b R_{bji}{}^a E_a, \\ {}^S\nabla_{E_{\bar{j}}} E_{\bar{i}} &= 0 \end{aligned}$$

with respect to the adapted frame $\{E_\lambda\}$, where Γ^h_{ji} denote the Christoffel symbols constructed with g_{ji} on M [12,16] (*for details*, *see* [21, *p*. 160]).

Let us consider a tensor field $\tilde{J}$ of type (1,1) on $T(M)$ defined by

$$\tilde{J}{}^H X = {}^V X, \tilde{J}{}^V X = -{}^H X$$

for any $X \in \Im^1_0(M)$, i.e., $\tilde{J} E_i = E_{\bar{i}}$, $\tilde{J} E_{\bar{i}} = -E_i$. Then we obtain $\tilde{J}^2 = -I$. Therefore, $\tilde{J}$ is an almost complex structure on $T(M)$. This almost complex structure is called an *adapted almost complex structure*. It is known that $\tilde{J}$ is integrable if and only if M is locally flat [6] (see also [21, p. 118]).

3. MAIN RESULTS

Let ∇ be an affine connection on M. A vector field V on M is called an *infinitesimal projective transformation* if there exists a 1-form Ω on M such that

$$(L_V\nabla)(X,Y)=\Omega(X)Y+\Omega(Y)X$$

for any $X,Y\in\Im_0^1(M)$. Next let (M,J,g) be an almost complex manifold with an affine connection ∇. A vector field V on M is called an *infinitesimal holomorphically projective transformation* if there exists a 1-form Ω on M such that

$$(L_V\nabla)(X,Y)=\Omega(X)Y+\Omega(Y)X-\Omega(JX)JY-\Omega(JY)JX$$

for any $X,Y\in\Im_0^1(M)$, where L_V is the Lie derivation with respect to V. In this case Ω is also called the *associated 1-form* of V. Especially, if $\Omega=0$, then V is an infinitesimal affine transformation.

Theorem 4. *Let (M,g) be a Riemannian manifold and $T(M)$ its tangent bundle with the Sasaki metric and the adapted almost complex structure. A vector field $\tilde V$ is an infinitesimal holomorphically projective transformation with associated 1-form $\tilde\Omega$ on $T(M)$ if and only if there exists $\varphi,\ \psi\in\Im_0^0(M)$, $B=(B^h)$, $D=(D^h)\in\Im_0^1(M)$, $A=(A_i^h)$, $C=(C_i^h)\in\Im_1^1(M)$ satisfying*

1. $\begin{pmatrix}\tilde V^h\\ \tilde V^{\bar h}\end{pmatrix}=\begin{pmatrix}B^h+y^aA_a^h+2\varphi y^h-y^a\Psi_a y^h\\ D^h+y^aC_a^h+2\psi y^h+y^a\Phi_a y^h\end{pmatrix}$,
2. $(\tilde\Omega_i,\tilde\Omega_{\bar i})=(\partial_i\psi,\partial_i\varphi)=(\Psi_i,\Phi_i)$,
3. $\nabla_i\Phi_j=0,\nabla_i\Psi_j=0$,
4. $\nabla_iA_j^a=\Phi_j\delta_i^a-\Phi_i\delta_j^a-\frac{1}{2}D^hR_{hji}^a$,
5. $\nabla_iC_j^a=\Psi_j\delta_i^a-\Psi_i\delta_j^a-B^cR_{cij}^a$,
6. $L_B\Gamma_{ji}^a=\nabla_j\nabla_iB^a+R_{hji}^aB^h=\Psi_j\delta_i^a+\Psi_i\delta_j^a$,
7. $\nabla_j\nabla_iD^a=-\Phi_j\delta_i^a-\Phi_i\delta_j^a+\frac{1}{2}R_{jih}^aD^h$,
8. $A_b^hR_{hij}^a=-2\varphi R_{bij}^a, C_b^hR_{hji}^a=-2\psi R_{bji}^a$,
9. $\Phi_lR_{kji}^a=0,\Psi_lR_{kji}^a=0$,
10. $B^h\nabla_hR_{bji}^a=R_{bji}^h\nabla_hB^a+R_{bjh}^a\nabla_iB^h$,
11. $R_{bji}^h\nabla_hD^a=0$,
12. $D^h\nabla_jR_{hbi}^a=R_{bhj}^a\nabla_iD^h+2R_{bhi}^a\nabla_jD^h$,
13. $B^h\nabla_hR_{jib}^a=R_{jib}^hC_h^a-R_{jih}^aC_b^h-R_{jhb}^a\nabla_iB^h-R_{hib}^a\nabla_jB^h$,

where $\tilde V=\begin{pmatrix}\tilde V^h\\ \tilde V^{\bar h}\end{pmatrix}=\tilde V^aE_a+\tilde V^{\bar a}E_{\bar a}$ *and* $\tilde\Omega=(\tilde\Omega_i,\tilde\Omega_{\bar i})=\tilde\Omega_a dx^a+\tilde\Omega_{\bar a}\delta y^a$.

Proof. We shall use the method proposed by Hasegawa and Yamauchi [9,10] in the proof and prove only the necessary condition because it is easy to prove the sufficient condition.

Let $\tilde V$ be an infinitesimal holomorphically projective transformation with the associated 1-form $\tilde\Omega$ on $T(M)$

$$(L_{\tilde V}{}^S\nabla)(\tilde X,\tilde Y)=\tilde\Omega(\tilde X)\tilde Y+\tilde\Omega(\tilde Y)\tilde X-\tilde\Omega(\tilde J\tilde X)\tilde J\tilde Y-\tilde\Omega(\tilde J\tilde Y)\tilde J\tilde X \tag{5}$$

for any $\tilde{X},\tilde{Y}\in\Im_0^1(T(M))$. From $(L_{\tilde{V}}{}^S\nabla)(E_{\bar{j}},E_{\bar{i}})=\tilde{\Omega}_{\bar{j}}E_{\bar{i}}+\tilde{\Omega}_{\bar{i}}E_{\bar{j}}-\tilde{\Omega}_jE_i-\tilde{\Omega}_iE_j$, we obtain

$$\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^h=-\tilde{\Omega}_j\delta_i^h-\tilde{\Omega}_i\delta_j^h \tag{6}$$

and

$$\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^{\bar{h}}=\tilde{\Omega}_{\bar{j}}\delta_i^h+\tilde{\Omega}_{\bar{i}}\delta_j^h. \tag{7}$$

Contracting i and h in (6), we have

$$\tilde{\Omega}_j=\partial_{\bar{j}}\tilde{\psi},$$

where $\tilde{\psi}=-\frac{1}{n+1}\partial_{\bar{a}}\tilde{V}^a$. Hence (6) is rewritten as follows:

$$\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^h=-(\partial_{\bar{j}}\tilde{\psi})\delta_i^h-(\partial_{\bar{i}}\tilde{\psi})\delta_j^h. \tag{8}$$

Differentiating (8) partially, we have

$$\begin{aligned}\partial_{\bar{k}}\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^h&=-(\partial_{\bar{k}}\partial_{\bar{j}}\tilde{\psi})\delta_i^h-(\partial_{\bar{k}}\partial_{\bar{i}}\tilde{\psi})\delta_j^h\\&=-(\partial_{\bar{j}}\partial_{\bar{i}}\tilde{\psi})\delta_k^h-(\partial_{\bar{i}}\partial_{\bar{k}}\tilde{\psi})\delta_i^h\\&=-(\partial_{\bar{i}}\partial_{\bar{k}}\tilde{\psi})\delta_j^h-(\partial_{\bar{i}}\partial_{\bar{j}}\tilde{\psi})\delta_k^h,\end{aligned}$$

from which we obtain

$$\partial_{\bar{k}}\partial_{\bar{j}}(\partial_{\bar{i}}\tilde{V}^h+2\tilde{\psi}\delta_i^h)=0.$$

Therefore we can put

$$P_{ji}{}^h=\partial_{\bar{j}}(\partial_{\bar{i}}\tilde{V}^h+2\tilde{\psi}\delta_i^h) \tag{9}$$

and

$$A_i^h+y^aP_{ai}{}^h=\partial_{\bar{i}}\tilde{V}^h+2\tilde{\psi}\delta_i^h, \tag{10}$$

where A_i^h and $P_{ji}{}^h$ are certain functions which depend only on the variables (x^h). The coordinate transformation rule implies that $A=(A_i^h)\in\Im_1^1(M)$ and $P=(P_{ji}{}^h)\in\Im_2^1(M)$. Using (8), we have

$$P_{ji}{}^h+P_{ij}{}^h=2\left\{\partial_{\bar{j}}\partial_{\bar{i}}\tilde{V}^h+(\partial_{\bar{j}}\tilde{\psi})\delta_i^h+(\partial_{\bar{i}}\tilde{\psi})\delta_j^h\right\}=0,$$

from which, using (9), we obtain

$$P_{ji}{}^h=\frac{1}{2}(P_{ji}{}^h-P_{ij}{}^h)=(\partial_{\bar{j}}\tilde{\psi})\delta_i^h-(\partial_{\bar{i}}\tilde{\psi})\delta_j^h. \tag{11}$$

On the other hand, using (10), we have

$$\tilde{\psi}=-\varphi+y^a\Psi_a,$$

where $\varphi=-\frac{1}{n-1}A_a^a$ and $\Psi_i=\frac{1}{n-1}P_{ia}{}^a$, from which

$$\tilde{\Omega}_i=\partial_{\bar{i}}\tilde{\psi}=\Psi_i. \tag{12}$$

Using (10), (11), and (12), we obtain

$$\partial_{\bar{i}}\tilde{V}^h=A_i^h+2\varphi\delta_i^h-y^h\Psi_i-y^a\Psi_a\delta_i^h,$$

from which

$$\tilde{V}^h=B^h+y^aA_a^h+2\varphi y^h-y^a\Psi_ay^h,$$

where B^h are certain functions which depend only on (x^h). The coordinate transformation rule implies that $B=(B^h)\in\Im_0^1(M)$.

Similarly, from (7), there exists $\psi \in \Im_0^0(M)$, $\Phi = (\Phi_i) \in \Im_1^0(M)$, $D = (D^h) \in \Im_0^1(M)$, and $C = (C_i^h) \in \Im_1^1(M)$ satisfying

$$\tilde{\varphi} = \psi + y^a \Phi_a,$$

$$\tilde{\Omega}_{\bar{i}} = \partial_{\bar{i}} \tilde{\varphi} = \Phi_i$$

and

$$\tilde{V}^{\bar{h}} = D^h + y^a C_a^h + 2\psi y^h + y^a \Phi_a y^h,$$

where $\tilde{\varphi} = \frac{1}{n+1} \partial_{\bar{a}} \tilde{V}^{\bar{a}}$ and $\psi = -\frac{1}{n-1} C_a^a$.

Next, from (5), we have

$$(L_{\tilde{V}} {}^S\nabla)(E_{\bar{j}}, E_i) = \Phi_j E_i + \Phi_i E_j + \Psi_j E_{\bar{i}} + \Psi_i E_{\bar{j}} \tag{13}$$

or

$$(L_{\tilde{V}} {}^S\nabla)(E_j, E_{\bar{i}}) = \Phi_j E_i + \Phi_i E_j + \Psi_j E_{\bar{i}} + \Psi_i E_j.$$

From (13), we get

$$\begin{aligned}
& (\Phi_j \delta_i^a + \Phi_i \delta_j^a) E_a + (\Psi_j \delta_i^a + \Psi_i \delta_j^a) E_{\bar{a}} \\
= & \left\{ \left(\nabla_i A_j^a + 2\partial_i \varphi \delta_j^a + \frac{1}{2} D^h R_{hji}^a \right) + y^b \left(-\delta_b^a \nabla_i \Psi_j - \delta_j^a \nabla_i \Psi_b \right. \right. \\
& + \frac{1}{2} B^h \nabla_h R_{bji}{}^a + \frac{1}{2} C_b^h R_{hji}{}^a + 2\psi R_{bji}{}^a + \frac{1}{2} C_j^h R_{bhi}{}^a \\
& \left. - \frac{1}{2} R_{bji}^h \nabla_h B^a - \frac{1}{2} R_{bjh}^a \nabla_i B^h \right) + y^b y^c \left(\frac{1}{2} \Phi_i R_{jbc}^a + 2\Phi_c R_{jbi}^a \right) \\
& \left. + y^b y^c y^h \left(\frac{1}{2} R_{bjh}^a \nabla_i \Psi_c + \frac{1}{2} R_{bji}^a \nabla_h \Psi_c - \frac{1}{2} \Psi_c \nabla_h R_{bji}^a \right) \right\} E_a \\
& + \left\{ (\nabla_i C_j^a + 2\partial_i \psi \delta_j^a + B^c R_{cij}^a) + y^b \left(\delta_a^b \nabla_i \Phi_j + \delta_j^a \nabla_i \Phi_b \right. \right. \\
& \left. \left. + A_b^c R_{cij}{}^a + 2\varphi R_{bij}{}^a + \frac{1}{2} A_j^c R_{cib}{}^a + \varphi R_{jib}{}^a - \frac{1}{2} R_{bji}^h \nabla_h D^a \right) + y^b y^c (\Psi_b R_{icj}^a + \Psi_c R_{jbi}^a) \right\} \\
& + \frac{1}{2} \left(\Psi_j R_{icb}^a + y^b y^c y^h \left(\frac{1}{2} R_{jbi}^a \; \nabla_h \Phi_c \right) \right) E_{\bar{a}}.
\end{aligned}$$

Comparing both sides of the above equation, we obtain

$$\Phi_j = \partial_j \varphi, \nabla_i \Phi_j = 0,$$

$$\Psi_j = \partial_j \psi, \nabla_i \Psi_j = 0,$$

$$\nabla_i A_j^a = \Phi_j \delta_i^a - \Phi_i \delta_j^a - \frac{1}{2} D^h R_{hji}^a,$$

$$\nabla_i C_j^a = \Psi_j \delta_i^a - \Psi_i \delta_j^a - B^c R_{cij}^a,$$

$$C_b^h R_{hji}^a = -2\psi R_{bji}^a, \Psi_l R_{kji}^a = 0,$$

$$A_b^h R_{hij}^a = -2\varphi R_{bij}^a, \Phi_l R_{kji}^a = 0,$$

$$B^h\nabla_h R^a_{bji} = R^h_{bji}\nabla_h B^a + R^a_{bjh}\nabla_i B^h, R^h_{bji}\nabla_h D^a = 0.$$

Lastly, from $(L_{\tilde{V}}{}^S\nabla)(E_j,E_i) = \Psi_j E_i + \Psi_i E_j - \Phi_j E_{\bar{i}} - \Phi_i E_{\bar{j}}$, we obtain

$$(\Psi_j\delta^a_i + \Psi_i\delta^a_j)E_a - (\Phi_j\delta^a_i + \Phi_i\delta^a_j)E_{\bar{a}} = \Bigg\{(L_B\Gamma^a_{ji}) + y^b\Bigg(A^h_b R^a_{hji} + 2\varphi R^a_{bji} + \frac{1}{2}A^a_h R^h_{jib} + \varphi R^a_{jib}$$

$$-\frac{1}{2}D^h\nabla_j R^a_{hbi} - \frac{1}{2}R^a_{hbi}\nabla_j D^h + \frac{1}{2}R^a_{bhj}\nabla_i D^h + \frac{1}{2}R^a_{bhi}\nabla_j D^h\Bigg)\Bigg\}E_a + \Bigg\{\Bigg(\nabla_j\nabla_i D^a - \frac{1}{2}R^a_{jih}D^h\Bigg)$$

$$+y^b\Bigg(-\frac{1}{2}B^h\nabla_h R^a_{jib} - \frac{1}{2}R^a_{jih}C^h_b + \frac{1}{2}R^h_{jib}C^a_h - \frac{1}{2}R^a_{jhb}\nabla_i B^h - \frac{1}{2}R^a_{hib}\nabla_j B^h\Bigg)\Bigg\}E_{\bar{a}},$$

from which we get the following important information:

$$L_B\Gamma^a_{ji} = \nabla_j\nabla_i B^a + R^a_{hji}B^h = \Psi_j\delta^a_i + \Psi_i\delta^a_j, \tag{14}$$

$$\nabla_j\nabla_i D^a = -\Phi_j\delta^a_i - \Phi_i\delta^a_j + \frac{1}{2}R^a_{jih}D^h,$$

$$D^h\nabla_j R^a_{hbi} = R^a_{bhj}\nabla_i D^h + 2R^a_{bhi}\nabla_j D^h,$$

$$B^h\nabla_h R^a_{jib} = R^h_{jib}C^a_h - R^a_{jih}C^h_b - R^a_{jhb}\nabla_i B^h - R^a_{hib}\nabla_j B^h.$$

From (14), we also note that B is an infinitesimal projective transformation on M. This completes the proof. □

Using Theorem 4, we at last come to the following:

Theorem 5. *Let (M,g) be a Riemannian manifold and $T(M)$ its tangent bundle with the Sasaki metric and the adapted almost complex structure. If $T(M)$ admits a non-affine infinitesimal holomorphically projective transformation, then M is locally flat.*

Proof. Let $\tilde{V}$ be a non-affine infinitesimal holomorphically projective transformation on M. Using 3. in Theorem 4, we have $\nabla_i\|\Phi\|^2 = \nabla_i\|\Psi\|^2 = 0$. Hence, $\|\Phi\|$ and $\|\Psi\|$ are constant on M. Suppose that M is not locally flat, then $\Phi = \Psi = 0$ by virtue of 9. in Theorem 4, that is, $\tilde{V}$ is an infinitesimal affine transformation. This is a contradiction. Therefore, M is locally flat. In this case, $T(M)$ is locally flat [7,12,20], [21, p. 166]. □

4. CONCLUSIONS

Tangent bundles of differentiable manifolds are of great importance in many areas of mathematics and physics. In the last decades a large number of publications have been devoted to the study of their special differential geometric properties.

Let (M,g) be a Riemannian manifold. A Riemannian metric $\tilde{g}$ on the tangent bundle $T(M)$ of M is said to be *natural* with respect to g on M if

$$\begin{aligned} \text{i)}\ \tilde{g}({}^HX,{}^HY) &= g(X,Y),\\ \text{ii)}\ \tilde{g}({}^HX,{}^VY) &= 0 \end{aligned}$$

for all vector fields $X,Y \in \Im^1_0(M)$. A natural metric $\tilde{g}$ is constructed in such a way that the vertical and horizontal subbundles are orthogonal and the bundle map π: $(T(M),\tilde{g}) \to (M,g)$ is a Riemannian submersion. All the preceding metrics belong to the wide class of the so-called *g-natural metrics* on the tangent bundle, initially classified by Kowalski and Sekizawa [13] and fully characterized by Abbassi and Sarih [1–3]. A well-known example of g-natural metrics is the Sasaki metric ^{S}g. Its construction is based on a natural splitting of the tangent bundle TTM of $T(M)$ into its vertical and horizontal subbundles by means of the Levi-Civita connection ∇ on (M,g). The purpose of this article is to characterize infinitesimal holomorphically projective transformations with respect to the Sasaki metric. In Theorem 4 we give a necessary and a sufficient condition for the vector field $\tilde{V}$ on $(T(M), {}^{S}\nabla, \tilde{J})$ to be an infinitesimal holomorphically projective transformation. This condition is represented by a set of relations involving certain tensor fields on M of type $(0,0)$, $(1,0)$, and $(1,1)$. We obtain these relations by giving the formula (5) in an adapted frame. Further, in Theorem 5 we show that if $(T(M), {}^{S}\nabla, \tilde{J})$ has a non-affine infinitesimal holomorphically projective transformation, then M and $T(M)$ are locally flat. Another well-known g-natural Riemannian metric g_{CG} had been defined, some years before, by Musso and Tricerri [15] who, inspired by paper [5] of Cheeger and Gromoll, called it the *Cheeger–Gromoll metric*. The Levi-Civita connection of g_{CG} and its Riemannian curvature tensor are calculated by Sekizawa in [17] (for more details, see [7]). Similar problems may be investigated for the Cheeger–Gromoll metric.

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Infinitesimaalsetest holomorfselt projektiivsetest teisendustest Sasaki meetrika suhtes puutujavektorkondadel

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Artikli eesmärgiks on leida lahendeid osatuletistega diferentsiaalvõrrandite süsteemile, mis kirjeldavad infinitesimaalseid holomorfselt projektiivseid teisendusi puutujavektorkonnal Sasaki meetrikaga ja vastavat adapteeritud peaaegu komplieksset struktuuri. On tõestatud, et juhul kui Riemanni muutkonna puutujavektorkonna struktuur lubab mitteafiinseid infinitesimaalseid holomorfselt projektiivseid teisendusi, siis Riemanni muutkond on lokaalselt tasane.