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MATHEMATICS

# Properties of $TQ$-algebras

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**Abstract.** Several properties of unital left (right) $TQ$-algebras are described. The conditions when a unital semitopological algebra is a left (right) $TQ$-algebra are given. It is shown that the space $\mathfrak{M}(A)$ (of nontrivial continuous multiplicative linear functionals on $A$) in the Gelfand topology is a compact Hausdorff space for every unital $TQ$-algebra with a nonempty set $\mathfrak{M}(A)$ and a commutative complete metrizable unital algebra is a $TQ$-algebra if and only if all maximal topological ideals of $A$ are closed. Examples of $TQ$-algebras are given. Open problems are presented.

**Key words:** topological algebras, $Q$-algebras, $TQ$-algebras, topological ideals.

## 1. INTRODUCTION

1. Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ of a real number or $\mathbb{C}$ of complex numbers and $A$ a topological algebra over $\mathbb{K}$ with separately continuous multiplication and with the unit element $e_A$ (in short, a semitopological algebra).

An element $a \in A$ is *topologically left* (*right*) *invertible* in $A$ if[1] $e_A \in \overline{Aa}$ (respectively, $e_A \in \overline{aA}$), or equivalently, there exists a net $(a_\lambda)_{\lambda\in\Lambda}$ of elements of $A$ (*the topological left* (respectively, *right*) *inverse* for $a$) such that $(a_\lambda a)_{\lambda\in\Lambda}$ (respectively, $(aa_\lambda)_{\lambda\in\Lambda}$) converges to $e_A$ in $A$. We will denote by $G_l^t(A)$ (respectively, by $G_r^t(A)$) the set of all topologically left (right) invertible elements in $A$ and by $G_l^{tb}(A)$ (respectively, by $G_r^{tb}(A)$) the set of all elements in $G_l^t(A)$ (in $G_r^t(A)$) for which there exists a topological left (respectively right) inverse that is bounded.

Moreover, let $G^t(A) = G_l^t(A) \cap G_r^t(A)$, $G^{tb}(A) = G_l^{tb}(A) \cap G_r^{tb}(A)$ and $\mathfrak{G}^t(A)$ be the set of all elements $a \in A$ for which there is a net $(a_\lambda)_{\lambda\in\Lambda}$ of elements of $A$ such that[2] $(a_\lambda a)_{\lambda\in\Lambda}$ and $(aa_\lambda)_{\lambda\in\Lambda}$ converge to $e_A$ in $A$. It is clear that $\mathfrak{G}^t(A) \subset G^t(A)$ and $G^{tb}(A) \subset \mathfrak{G}^t(A)$ for complete pseudoconvex algebra $A$ (see [11], Theorem 3), but it is not known whether $\mathfrak{G}^t(A) \neq G^t(A)$ in general.

In addition, let $G_l(A)$ ($G_r(A)$) denote the set of all left (respectively, right) invertible elements in $A$ and $G(A) = G_l(A) \cap G_r(A)$. Then $G(A) \subset \mathfrak{G}^t(A) \subset G^t(A)$. In particular, when $G(A) = G^t(A)$, $A$ is called an *invertive algebra*[3] (see [2], p. 14) and a topological invertible element is said to be *proper* (see [34], p. 323) if it is non-invertible. Properties of topologically invertible elements have been discussed in several papers, for example, in [2], [6], [8], [9], [11], [12], [15], [17], [19], [25], and [31]–[34].

We shall say that a semitopological algebra $A$ is a *left* (*right*) $TQ$-*algebra* if $G_l^t(A)$ (respectively, $G_r^t(A)$) is open in $A$, a $TQ$-*algebra* if both sets $G_t^l(A)$ and $G_t^r(A)$ are open in $A$, a $\mathfrak{TQ}$-*algebra* if $\mathfrak{G}^t(A)$ is open in $A$, and a $Q$-*algebra* if the set $G(A)$ is open in $A$. It is easy to see that every invertive $TQ$-algebra is a $Q$-algebra but there exist $TQ$-algebras which are not $Q$-algebras (see Examples (c) and (e)). Call a $TQ$-algebra proper if it is not a $Q$-algebra.

2. Let $A$ be a unital semitopological algebra, $m(A)$ the set of all closed two-sided ideals in $A$, which are

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[1] Here and later on every $\overline{U}$ denotes the closure of $U$ in $A$.

[2] That is, the net $(a_\lambda)_{\lambda\in\Lambda}$ is the same in $(a_\lambda a)_{\lambda\in\Lambda}$ and $(aa_\lambda)_{\lambda\in\Lambda}$.

[3] It is known (see [2], Corollary 2) that every complete unital locally $m$-pseudoconvex algebra is an invertive algebra, but every commutative complete metrizable unital algebra with a discontinuous inverse is not (see [32], Proposition 4).

maximal as left or right ideals in $A$. A semitopological algebra $A$ over $\mathbb{K}$ is called a *Gelfand–Mazur algebra* if $A/M$ (in the quotient topology) is topologically isomorphic to $\mathbb{K}$ for each $M \in m(A)$, and a *simplicial algebra* if every closed left (right) ideal of $A$ is contained in some closed maximal left (respectively, right) ideal of $A$. The main classes of Gelfand–Mazur algebras have been described in [1] and [4]. It is known (see[4] [5], Theorem 4) that every commutative unital locally $m$-pseudoconvex[5] Hausdorff algebra is simplicial.

3. Let $A$ be a semitopological algebra with a unit element $e_A$, $\mathfrak{M}(A)$ the set of all nontrivial continuous multiplicative linear functionals on $A$,

$$\sigma_l^t(x) = \{\lambda \in \mathbb{C} : x - \lambda e_A \notin G_l^t(A)\}$$

$$(\sigma_r^t(x) = \{\lambda \in \mathbb{C} : x - \lambda e_A \notin G_r^t(A)\})$$

the *left* (respectively, *right*) *topological spectrum* of $x \in A$, and

$$\rho_l^t(x) = \sup\{|\lambda| : \lambda \in \sigma_l^t(x)\}$$

$$(\text{respectively, } \rho_r^t(x) = \sup\{|\lambda| : \lambda \in \sigma_r^t(x)\})$$

the *left* (respectively, *right*) *topological spectral radius* of $x \in A$. Then the *topological spectrum*

$$\sigma^t(x) = \{\lambda \in \mathbb{C} : x - \lambda e_A \notin G^t(A)\}$$

of $x \in A$, described in [6], coincides with the set $\sigma_l^t(x) \cup \sigma_r^t(x)$ in $\mathbb{C}$, and the *topological spectral radius*

$$\rho^t(x) = \sup\{|\lambda| : \lambda \in \sigma^t(x)\}$$

of $x \in A$ is equal to $\max\{\rho_l^t(x), \rho_r^t(x)\}$. If $\mathfrak{M}(A)$ is not empty, then

$$\{\varphi(x) : \varphi \in \mathfrak{M}(A)\} \subset \sigma^t(x)$$

for each $x \in A$. In particular, when

$$\{\varphi(x) : \varphi \in \mathfrak{M}(A)\} = \sigma^t(x)$$

for each $x \in A$, we say that $A$ has the *functional topological spectrum*. In this case

$$\rho^t(x) = \sup\{|\varphi(x)| : \varphi \in \sigma^t(x)\}$$

for each $x \in A$.

4. Several properties of unital $TQ$-algebras are presented in the present paper. The conditions when a unital semitopological algebra is a $TQ$-algebra are given. It is shown that the space $\mathfrak{M}(A)$ in the Gelfand topology is a compact Hausdorff space for every unital $TQ$-algebra with a nonempty set $\mathfrak{M}(A)$, and a commutative complete metrizable unital algebra is a $TQ$-algebra if and only if all maximal topological ideals of $A$ are closed. In addition, examples of $TQ$-algebras are given and several open problems are presented.

## 2. EXAMPLES OF $TQ$-ALGEBRAS

Now we give some examples of $TQ$-algebras.

(a) **Strongly sequential algebras.** Every unital (commutative or not) normed algebra (similarly, every unital $p$-normed algebra with $p \in (0,1]$) is a $\mathfrak{TQ}$-algebra, hence also a $TQ$-algebra (see [12], Proposition 2.6).

More generally, every strongly sequential algebra[6] $A$ is a $\mathfrak{TQ}$-algebra. Indeed, let $U$ be a neighbourhood of zero in $A$ such that for each $x \in U$ the sequence $(x^n)$ converges to zero and let $x_0 \in U$ be an arbitrary element. Since

$$\Big(\sum_{k=0}^{n} x_0^k\Big)(e_A - x_0) - e_A = (e_A - x_0)\Big(\sum_{k=0}^{n} x_0^k\Big) - e_A = -x_0^{n+1} \tag{1}$$

and

$$\lim_{n\to\infty} x_0^n = \theta_A,$$

$e_A - U \in \mathfrak{G}_t(A)$. Hence, $A$ is a $\mathfrak{TQ}$-algebra and a $TQ$-algebra by Corollary 2 below.

(b) **Metrizable pseudo-Banach algebras**[7]. It is known (see [27], Proposition 4.6) that every metrizable pseudo-Banach locally convex algebra is a strongly sequential algebra. Hence, every such topological algebra is a $\mathfrak{TQ}$-algebra and thus a $TQ$-algebra as well.

(c) **The algebra** $(P(t);\ \tau_c)$. Let $P(t)$ be the algebra of all complex polynomials in one variable and $\tau_c$ for each $c \geqslant 1$ the locally convex topology on $P(t)$ described in [30]. All algebraic operations in $P(t)$ are defined pointwise. Then $(P(t), \tau_c)$ is a commutative unital locally convex (not normed) $TQ$-algebra (see [12], Example 2).

(d) **The algebra $P(\mathbb{T})$.** Let $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $P(\mathbb{T})$ be the unital algebra (with pointwise algebraic operations) of all polynomials on $\mathbb{T}$ with complex coefficients endowed with the uniform norm topology. Then $P(\mathbb{T})$ is an incomplete normed algeba which is not a $Q$-algebra (see [20], p. 73, or [21], p. 50). Hence, $P(\mathbb{T})$ is a $TQ$-algebra which is not a $Q$-algebra.

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4 For complete algebras see [3], Proposition 2, or [14], Corollary 7.1.14, and for locally $m$-convex algebras see [16], pp. 321–322.

5 A semitopological algebra $A$ is *locally m-pseudoconvex* if its topology is given by a family of nonhomogeneous submultiplicative seminorms (see, for example, [1] or [4]). When every seminorm in this family is homogeneous, then $A$ is a *locally m-convex algebra*.

6 A topological algebra $A$ is *strongly sequential* (see, for example, [24], p. 51) if there exists a neighbourhood $U$ of zero in $A$ such that for each $x \in U$ the sequence $(x^n)$ converges in $A$ to zero. It is known (see [24], Theorem 3.10) that a locally $m$-convex Fréchet algebra is strongly sequential if and only if it is a $Q$ algebra. Examples of strongly sequential algebras can be found in [27].

7 The notion of pseudo-Banach algebras was introduced in [10], p. 56.

(e) **Topologically simple algebras.** A commutative semitopological algebra is *topologically simple* if it has no closed proper non-zero ideals. Examples of commutative unital complete non-metrizable locally convex topologically simple Hausdorff algebras have been given in [13] and in [31]. It is shown in [31], Proposition 1, that every commutative unital topological algebra is topologically simple if and only if $G_t(A) = A \setminus \{\theta_A\}$. Hence, topological algebras in [13] and in [31] described above are commutative $TQ$-algebras.

## 3. PROPERTIES OF $TQ$-ALGEBRAS

We shall show that unital $TQ$-algebras are very similar to unital $Q$-algebras. First we prove the following

**Proposition 1.** *Let $A$ be a unital left (right or two-sided) $TQ$-algebra. Then*

$$A = \{x \in A : \rho_l^t(x) < \infty\}$$

$$(\text{respectively}, A = \{x \in A : \rho_r^t(x) < \infty\}$$

*and*

$$A = \{x \in A : \rho^t(x) < \infty\}).$$

*Proof.* Let $A$ be a unital left $TQ$-algebra. Then $G_l^t(A)$ is a neighbourhood of $e_A$ in $A$. Therefore, there is a balanced neighbourhood $U$ of zero in $A$ such that $e_A + U \subset G_l^t(A)$ and for each $x \in A$ there is an $\varepsilon_x > 0$ such that $\lambda x \in U$ whenever $|\lambda| \leqslant \varepsilon_x$. Let $\mu \in (0, \varepsilon_x)$ be a fixed number. Suppose that $\rho_l^t(x) > \frac{1}{\mu}$. Then there is a number $\lambda \in \sigma_l^t(x)$ such that $|\lambda| > \frac{1}{\mu}$. On the other hand, since $|\frac{1}{\lambda\mu}| < 1$, then

$$x - \lambda e_A = -\lambda\Big(e_A - \frac{1}{\lambda}x\Big) = -\lambda\Big(e_A + \Big(-\frac{1}{\lambda\mu}\Big)\mu x\Big)$$
$$\in -\lambda(e_A + U) \subset -\lambda G_l^t(A) \subset G_l^t(A).$$

This means that $\lambda \notin \sigma_l^t(x)$. The condition shows that $\rho_l^t(x) \leqslant \frac{1}{\mu} < \infty$ for each $x \in A$.

Proofs for other cases are similar. □

**Corollary 1.** *Let $A$ be a left (right) $TQ$-algebra. Then $\sigma_l^t(x)$, $\sigma_r^t(x)$, and $\sigma^t(x)$ are compact subsets in $\mathbb{C}$ for each $x \in A$.*

*Proof.* Let $A$ be a left $TQ$-algebra, $x \in A$, and $\lambda_0 \in \mathbb{C} \setminus \sigma_l^t(x)$. Then $x - \lambda_0 e_A \in G_l^t(A)$. Since the map $\lambda \mapsto x - \lambda e_A$ is continuous at $\lambda_0$ and $G_l^t(A)$ is a neighbourhood of $x - \lambda_0 e_A$, then there is a neighbourhood $O(\lambda_0)$ of $\lambda_0$ in $\mathbb{C}$ such that $x - \lambda e_A \in G_l^t(A)$ for each $\lambda \in O(\lambda_0)$. Hence $O(\lambda_0) \subset \mathbb{C} \setminus \sigma_l^t(x)$. Therefore, $\sigma_l^t(x)$ is a closed set in $\mathbb{C}$. By Proposition 1, $\sigma_l^t(x)$ is a compact subset of $\mathbb{C}$.

Proofs for a right $TQ$-algebra and a $TQ$-algebra are similar. □

**Theorem 1.** *Let $A$ be a unital semitopological algebra. Then the following statements are equivalent*:

(a) *$A$ is a left (right) $TQ$-algebra*;

(b) *the set $G_l^t(A)$ (respectively, $G_r^t(A)$) is a neighbourhood of $e_A$ in $A$*;

(c) *$e_A$ is an interior point of $G_l^t(A)$ (respectively, $G_r^t(A)$)*;

(d) *the interior of $G_l^t(A)$ (respectively, the interior of $G_r^t(A)$) is not empty*;

(e) *$S_l(A) = \{x \in A : \rho_l^t(x) \leqslant 1\}$ (respectively, $S_r(A) = \{x \in A : \rho_r^t(x) \leqslant 1\}$) is a neighbourhood of zero in $A$*;

(f) *there is a balanced neighbourhood of zero $V$ in $A$ such that*[8] *$\rho_l^t(x) \leqslant g_V(x)$ (respectively, $\rho_r^t(x) \leqslant g_V(x)$) for each $x \in A$*;

(g) *the topological left spectral radius $\rho_l^t$ (respectively, right spectral radius $\rho_r^t$) is upper semi-continuous*;

(h) *the topological left spectral radius $\rho_l^t$ (respectively, right spectral radius $\rho_r^t$) is continuous at $\theta_A$.*

*Proof.* The implications (a) ⇒ (b) ⇒ (c) ⇒ (d) are trivial.

(d) ⇒ (a) There is a non-void open subset $U \subset G_l^t(A)$. Let $z_0 \in U$ and put

$$Z = \{z \in A : z_0 z \in U\}.$$

Then $e \in Z$ and, by the continuity of multiplication, there is an open neighbourhood $V$ of $e$ with $z_0 V \subset U$. Let $y \in V$ such that $z_0 y \in U \subset G_l^t(A)$. Then there is a net $(y_\lambda)_{\lambda\in\Lambda}$ in $A$ with $(y_\lambda z_0 y)_{\lambda\in\Lambda} \to e$. Thus $y \in G_l^t(A)$ and so $V \subset G_l^t(A)$.

Let now $x_0$ be an arbitrary element of $G_l^t(A)$. Then there is a net $(x_\lambda)_{\lambda\in\Lambda}$ such that $(x_\lambda x_0)_{\lambda\in\Lambda} \to e$, and so for some $\lambda_0$ we have $x_{\lambda_0} x_0 \in V$. Again, by the continuity of multiplication, there is an open neighbourhood $V_0$ of $x_0$ with $x_{\lambda_0} V_0 \subset V \subset G_l^t(A)$. It means that all elements in $V_0$ are topologically left invertible. Since the element $x_0$ was chosen arbitrarily, the set $G_l^t(A)$ is open and the implication follows.

(b) ⇒ (e) Let $G_l^t(A)$ be a neighbourhood of $e_A$ in $A$. Then there exists a balanced neighbourhood $U$ of zero such that $e_A + U \subset G_l^t(A)$. Suppose that there is an element $u \in U \setminus S_l(A)$. Then $\rho_l^t(u) > 1$. Therefore there is a number $\lambda \in \sigma_l^t(u)$ such that $|\lambda| > 1$. Hence $u - \lambda e_A \notin G_l^t(A)$. On the other hand, since $u \in U$, $U$ is balanced, and $|\frac{1}{\lambda}| < 1$, then

$$u - \lambda e_A = -\lambda\Big[e_A + \Big(-\frac{1}{\lambda}\Big)u\Big]$$
$$\in -\lambda(e_A + U) \subset -\lambda G_l^t(A) \subset G_l^t(A).$$

Therefore, $U \subset S_l(A)$. It means that $S_l(A)$ is a neighbourhood of $\theta_A$ in $A$.

(e) ⇒ (a) Let $A$ be a unital topological algebra such that $S_l(A)$ is a neighbourhood of zero in $A$. Suppose that

[8] Here and later on $g_V(x) = \inf\{\lambda > 0 : x \in \lambda V\}$ for each $x \in A$, that is, $g_V$ is the Minkowski functional of $V$.

there is an element $u \in S_l(A)$ such that $e_A + 2u \notin G_l^t(A)$. Then $2 \in \sigma_l^t(u)$ implies $\rho_l^t(u) \geqslant 2$, which is impossible. Hence, $e_A + 2S_l(A) \subset G_l^t(A)$. Consequently, $A$ is a left $TQ$-algebra.

(b) $\Rightarrow$ (f) Let $x \in A$ and $G_l^t(A)$ be a neighbourhood of $e_A$. Then, there is a balanced neighbourhood $V$ of zero such that $e_A + V \subset G_l^t(A)$. If $\mu$ is an arbitrary positive number such that $x \in \mu V$, then from $x - \mu e_A = -\mu(e_A - \frac{1}{\mu}x) \in G_l^t(A)$ it follows that $\mu \notin \sigma_l^t(x)$. Therefore $\rho_l^t(x) < \mu$ for each $\mu > 0$ such that $x \in \mu V$. Hence,

$$\rho_l^t(x) \leqslant \inf\{\lambda > 0 : u \in \lambda V\} = g_V(u).$$

(f) $\Rightarrow$ (b) Suppose that there is a balanced neighbourhood $V$ of zero in $A$ such that $\rho_l^t(x) \leqslant g_V(x)$ for each $x \in A$. If $x \in \frac{1}{2}V$ is an arbitrary element, then $\rho_l^t(x) \leqslant g_V(x) \leqslant \frac{1}{2} < 1$. Hence, $1 \notin \sigma_l^t(x)$. It follows from $e_A + \frac{1}{2}V \subset G_l^t(A)$ that $G_l^t(A)$ is a neighbourhood of $e_A$ in $A$.

(a) $\Rightarrow$ (g) Let $A$ be a left $TQ$-algebra. Then $\rho_l^t(x) < \infty$ for each $x \in A$ by Proposition 1. To show that the set $\{x \in A : \rho_l^t(x) \geqslant \alpha\}$ is closed in $A$ for each $\alpha \in \mathbb{R}$, it is enough to show that $\{x \in A : \rho_l^t(x) < \alpha\}$ is open in $A$ for each $\alpha \in (0,\infty)$ (because $\{x \in A : \rho_l^t(x) \geqslant \alpha\} = A$ if $\alpha \leqslant 0$). For this, let $\alpha_0 \in (0,\infty)$ and $x_0 \in A$ be such that $\rho_l^t(x_0) < \alpha_0$. Then there is a number $\beta \in \mathbb{R}$ such that $\rho_l^t(x_0) < \beta < \alpha_0$.

Let $\Phi : A \times \mathbb{K} \to A$ be a map defined by $\Phi(x,\mu) = x - \mu e_A$ for each $(x,\mu) \in A \times \mathbb{K}$, and $\Psi : A \times \mathbb{K} \to A$ be a map defined by $\Psi(x,\mu) = e_A - \mu x$ for each $(x,\mu) \in A \times \mathbb{K}$. Since $\Phi$ and $\Psi$ are continuous maps, the sets $\Phi^{-1}(G_l^t(A))$ and $\Psi^{-1}(G_l^t(A))$ are open in $A \times \mathbb{K}$. Therefore from $(x_0,0) \in \Psi^{-1}(G_l^t(A))$ it follows that there exists a neighbourhood $O(x_0)$ of $x_0$ in $A$ and a neighbourhood $U$ of zero in $\mathbb{K}$ such that $O(x_0) \times U \subset \Psi^{-1}(G_l^t(A))$. Moreover, $U$ defines a number $M > 0$ such that $\mu^{-1} \in U$ whenever $|\mu| > M$. We can assume that $M > \beta$. Let $D = \{\nu \in \mathbb{K} : \beta \leqslant |\nu| \leqslant M\}$. Since $(x_0,\nu) \in \Phi^{-1}(G_l^t(A))$ for each $\nu \in D$, then for each fixed $\nu \in D$ there is a neighbourhood $O_\nu(x_0)$ of $x_0$ and an open neighbourhood $U(\nu)$ of $\nu$ in $\mathbb{K}$ such that $O_\nu(x_0) \times U(\nu) \subset \Phi^{-1}(G_l^t(A))$. It is clear that $D$ is a compact subset of $\mathbb{K}$. Therefore there exist $n \in \mathbb{N}$ and $\nu_1,\ldots,\nu_n \in D$ such that the sets $U(\nu_1),\ldots U(\nu_n)$ cover $D$. Let now

$$O'(x_0) = O(x_0) \cap \Big(\bigcap_{k=1}^{n} O_{\nu_k}(x_0)\Big).$$

Then $O'(x_0)$ is a neighbourhood of $x_0$ in $A$. If $|\alpha_0| > M$, then

$$(x, \alpha_0^{-1}) \in O(x_0) \times U \subset \Psi^{-1}(G_l^t(A))$$

or $x - \alpha_0 e_A \in G_l^t(A)$ for each $x \in O'(x_0)$. Moreover, if $\alpha_0 \in D$, then $\alpha_0 \in U(\nu_k)$ for some $k \in \{1,\ldots,n\}$. Since $(x,\alpha_0) \in O_{\nu_k}(x_0) \times U(\nu_k) \subset \Phi^{-1}(G_l^t(A))$, then $x - \alpha_0 e_A \in G_l^t(A)$ for each $x \in O'(x_0)$ as well. Hence, $\rho_l^t(x) < \alpha_0$ for each $x \in O'(x_0)$. Thus

$$O'(x_0) \subset \{x \in A : \rho_l^t(x) < \alpha_0\}.$$

It means that $\rho_l^t$ is upper semi-continuous.

(g) $\Rightarrow$ (h) Trivial because $\rho_l^t(\theta_A) = 0$.

(h) $\Rightarrow$ (d) By condition (g) it is clear that $U = \{x \in A : \rho_l^t(x) < 1\}$ is a neighbourhood of zero in A. Suppose that $e_A + U \not\subset G_l^t(A)$. Then there is an element $u_0 \in U$ such that $e_A + u_0 \notin G_l^t(A)$. Then $-1 \in \sigma_l^t(x)$, but this is impossible because $\rho_l^t(u_0) < 1$ . Therefore, $e_A + U \subset G_l^t(A)$. Consequently, $e_A$ is an interior point of $G_l^t(A)$.

The proof for a right $TQ$-algebra $A$ is similar. □

**Corollary 2.** *Let A be a unital semitopological algebra. Then the following statements are equivalent*:

(a) *$A$ is a $TQ$-algebra*;

(b) *the set $G^t(A)$ is a neighbourhood of $e_A$ in $A$*;

(c) *$e_A$ is an interior point of $G^t(A)$*;

(d) *the interior of $G^t(A)$ is not empty*;

(e) *$S^t(A) = \{x \in A : \rho^t(x) \leqslant 1\}$ is a neighbourhood of zero in $A$*;

(f) *there is a balanced neighbourhood of zero $V$ in $A$ such that $\rho^t(x) \leqslant g_V(a)$ for each $x \in A$*;

(g) *the topological spectral radius $\rho^t$ is upper semi-continuous*;

(h) *the topological spectral radius $\rho^t$ is continuous at $\theta_A$.*

**Theorem 2.** *Let $A$ be a unital m-barrelled semitopological algebra*[9]*with a nonempty set $\mathfrak{M}(A)$. If*

(1) *$A = \{x \in A : \rho^t(x) < \infty\}$*

*and*

(2) *$\sigma^t(x) = \{\varphi(x) : \varphi \in \mathfrak{M}(A)\}$ for each $x \in A$,*

*then $A$ is a $TQ$-algebra.*

*Proof.* Let $A$ be a unital semitopological $m$-barrelled algebra with nonempty $\mathfrak{M}(A)$, $x \in A$, and let

$$O_1 = \{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}$$

and

$$A_1 = \bigcap_{\varphi \in \mathfrak{M}(A)} \varphi^{-1}(O_1).$$

Then $A_1$ is a closed, idempotent, convex, and balanced set in $A$. Let

$$\delta = \sup_{\varphi \in \mathfrak{M}(A)} |\varphi(x)|.$$

Then $\delta \in \mathbb{R}$ by condition (1). If $\varphi(x) = 0$ for each $\varphi \in \mathfrak{M}(A)$, then $\lambda x \in A_1$ for each $\lambda \in \mathbb{R}$. If $\delta > 0$, then $\lambda x \in A_1$ whenever $|\lambda| \leqslant \frac{1}{\delta}$. Hence, $A_1$ is an absorbing set. Thus, $A_1$ is a neighbourhood of zero in $A$, because $A$ is $m$-barrelled. Since $A_1 \subset S^t(A)$ by assumption (2), then $S^t(A)$ is also a neighbourhood of zero in $A$. Consequently, $A$ is a unital $TQ$-algebra by Corollary 2. □

[9] That is, a semitopological algebra in which every closed, idempotent, convex, balanced, and absorbing subset is a neighbourhood of zero.

It is known (see [6], Corollaries 9 and 10) that every element in a commutative unital simplicial Gelfand–Mazur algebra (in particular, in a commutative unital locally $m$-pseudoconvex Hausdorff algebra) has a functional spectrum. Therefore, we have

**Corollary 3.** *Let $A$ be a commutative unital simplicial m-barrelled Gelfand–Mazur algebra such that $\mathfrak{M}(A)$ is nonempty* (*in particular, a commutative unital m-barrelled locally m-pseudoconvex Hausdorff algebra*). *If*

$$A = \{x \in A : \rho^t(x) < \infty\},$$

*then $A$ is a $TQ$-algebra.*

To describe the properties of the set $\mathfrak{M}(A)$ for a left (right) $TQ$-algebra $A$, we need the following result.

**Lemma 1.** (a) *Let $A$ be a unital semitopological algebra and let $x \in A$. If $\mathfrak{M}(A)$ is nonempty, then $\varphi(x) \neq 0$ for each $\varphi \in \mathfrak{M}(A)$ if $x \in G_l^t(A) \cup G_r^t(A)$ and for each*[10] *$\varphi \in \mathfrak{M}^\#(A)$ if $x \in G_l(A) \cup G_r(A)$.*

(b) *Let $A$ be a commutative unital simplicial Gelfand–Mazur algebra and $x \in A$. If $\varphi(x) \neq 0$ for each $\varphi \in \mathfrak{M}(A)$, then $x \in G^t(A)$.*

*Proof.* (a) Let $A$ be a topological algebra and $x \in G_l^t(A) \cup G_r^t(A)$. Then $x \in G_l^t(A)$ or $x \in G_r^t(A)$. Therefore there is a net $(x_\lambda)_{\lambda\in\Lambda}$ in $A$ such that $(x_\lambda x)_{\lambda\in\Lambda}$ converges to $e_A$ in $A$ or there is a net $(y_\mu)_{\mu\in M}$ in $A$ such that $(xy_\mu)_{\mu\in M}$ converges to $e_A$ in $A$. Hence $(\varphi(x_\lambda)\varphi(x))_{\lambda\in\Lambda}$ and $(\varphi(x)\varphi(y_\mu))_{\mu\in M}$ converge in $\mathbb{K}$ to 1 for each $\varphi \in \mathfrak{M}(A)$. Consequently, in both cases $\varphi(x) \neq 0$ for each $\varphi \in \mathfrak{M}(A)$.

Let now $x \in G_l(A) \cup G_r(A)$. Then $x \in G_l(A)$ or $x \in G_r(A)$. Therefore, there is an element $y \in A$ such that $yx = e_A$ or $xy = e_A$. Hence $\varphi(y)\varphi(x) = 1$ or $\varphi(x)\varphi(y) = 1$ for each $\varphi \in \mathfrak{M}^\#(A)$. Consequently, in both cases $\varphi(x) \neq 0$ for each $\varphi \in \mathfrak{M}^\#(A)$.

(b) See the proof of Proposition 8 in [6]. □

**Proposition 2.** *Let $A$ be a unital left (right) $TQ$-algebra. If $\mathfrak{M}(A)$ is not empty, then $\mathfrak{M}(A)$ is an equicontinuous subset of the topological dual space $A^*$ of $A$.*

*Proof.* Let $A$ be a left $TQ$-algebra. Then $G_l^t(A)$ is a neighbourhood of $e_A$ in $A$. Now there is a balanced neighbourhood $U$ of zero in $A$ such that $e_A + U \subset G_l^t(A)$. If[11] $\mathfrak{M}(A) \not\subset U^\circ$, then there are $\varphi_0 \in \mathfrak{M}(A)$ and $a_0 \in U$ such that $|\varphi_0(a_0)| > 1$. Let $\lambda_0 = \varphi_0(a_0)^{-1}$. Then $|\lambda_0| < 1$. Therefore $e_A - \lambda_0 a_0 \in e_A + U \subset G_l^t(A)$. On the other hand, it follows by Lemma 1(a) that $e_A - \lambda_0 a_0 \notin G_l^t(A)$. Hence, $\mathfrak{M}(A) \subset U^\circ$. Therefore $\mathfrak{M}(A)$ is an equicontinuous subset of $A^*$ by Proposition 6 in [23], p. 200.

The proof for a right $TQ$-algebra is similar. □

**Corollary 4.** *Let $A$ be a unital left (right) $TQ$-algebra. If $\mathfrak{M}(A)$ is not empty, then $\mathfrak{M}(A)$ is a compact Hausdorff space in the Gelfand topology.*

*Proof.* $\mathfrak{M}(A)$ is an equicontinuous subset of $A^*$ by Proposition 2. Therefore $\mathfrak{M}(A)$ is a relatively compact subset in the Gelfand topology by the Alaoglu–Bourbaki theorem. Since $\mathfrak{M}(A)$ is closed (because $A$ is a unital algebra[12]), then $\mathfrak{M}(A)$ is compact in the Gelfand topology. □

**Theorem 3.** *Let $A$ be a commutative unital simplicial Gelfand–Mazur algebra* (*in particular, a commutative unital locally m-pseudoconvex Hausdorff algebra*). *If $\mathfrak{M}(A)$ is equicontinuous, then $A$ is a left (right) $TQ$-algebra.*

*Proof.* Let $A$ be a commutative unital simplicial Gelfand–Mazur algebra such that $\mathfrak{M}(A)$ is equicontinuous. Then

$$\begin{aligned} U &= \{a \in A : |\varphi(a)| \leqslant 1 \text{ for each } \varphi \in \mathfrak{M}(A)\} \\ &= \bigcap_{\varphi\in\mathfrak{M}(A)} \varphi^{-1}(O_1) \end{aligned}$$

is a neighbourhood of zero in $A$ (see [29], p. 83, result 4.1). Therefore $V = \frac{1}{2}U$ is also a neighbourhood of zero in $A$. To show that $e_A + V \subset G_l^t(A)$, let $x \in e_A + V$. Then $|\varphi(x - e_A)| \leqslant \frac{1}{2}$ for each $\varphi \in \mathfrak{M}(A)$. Hence $\varphi(x) \neq 0$ for each $\varphi \in \mathfrak{M}(A)$. Therefore, $x \in G_l^t(A)$ by Lemma 1(b) and $e_A$ is an interior point of $G_l^t(A)$. We conclude that $A$ is a left $TQ$-algebra by Theorem 1. □

The following result shows that every non-invertive left (right) $TQ$-algebra has dense maximal ideals.

**Proposition 3.** (a) *Let $A$ be a unital semitopological algebra and $i_l(A)$ ($i_r(A)$) be the set of all closed left* (*respectively, right*) *ideals in $A$. Then*

$$G_l^t(A) = A \setminus \bigcup_{I\in i_l(A)} I \quad \text{and} \quad G_r^t(A) = A \setminus \bigcup_{I\in i_r(A)} I.$$

(b) *A unital left* (*right*) *$TQ$-algebra is a left* (*respectively, right*) *$Q$-algebra if and only if every maximal left* (*respectively, right*) *ideal of $A$ is closed.*

*Proof.* (a) Let $a \in G_l^t(A)$. Then there is a net $(a_\lambda)_{\lambda\in\Lambda}$ in $A$ such that $(a_\lambda a)_{\lambda\in\Lambda}$ converges to $e_A$. If $a$ belongs to some closed left ideal $I$ of $A$, then $a_\lambda a \in I$ for each $\lambda \in \Lambda$. Hence $e_A \in I$, but it is not possible. Consequently,

$$a \in A \setminus \bigcup_{I\in i_l(A)} I. \tag{2}$$

[10] Here and later on we denote by $\mathfrak{M}^\#(A)$ the set of all non-trivial (not necessarily continuous) multiplicative linear functionals on $A$.

[11] Here $U^\circ = \{\psi \in A^* : |\psi(a)| \leqslant 1 \text{ for each } a \in U\}$ is the polar of $U$.

[12] See, for example, [28], Theorem 11.9.

Let now $a \in A$ satisfy condition (2). If $a \notin G_l^t(A)$, then $a \notin G_l(A)$ (because $G_l(A) \subset G_l^t(A)$) and $Aa$ is a left ideal in $A$. Let $I$ denote the closure of $Aa$ in $A$. Then $I \neq A$ (because $a \notin G_l^t(A)$). Hence, $I \in i_l(A)$ and $a \in I$. By assumption it is not possible. Hence, $a \in G_l^t(A)$.

The proof for closed right ideals is similar.

(b) Let $A$ be a unital left $TQ$-algebra. If $A$ is a left $Q$-algebra, then every maximal left ideal in $A$ is closed. Vice versa, if every maximal left ideal of $A$ is closed, then

$$G_l(A) = A \setminus \bigcup_{M \in M_l(A)} M = A \setminus \bigcup_{M \in m_l(A)} M = G_l^t(A),$$

where $M_l(A)$ is the set of all maximal left ideals in $A$ and $m_l(A)$ is the subset of closed ideals in $M_l(A)$. Hence $A$ is a left $Q$-algebra.

The proof for right $TQ$ algebra is similar. □

**Proposition 4.** *Let $A$ be a unital semitopological Hausdorff algebra and $B$ a unital dense subalgebra of $A$ with the same unit element. Then*

$$G_l^t(B) = G_l^t(A) \cap B \ \ (\text{respectively} \ \ G_r^t(B) = G_r^t(A) \cap B).$$

*Proof.* It is clear that $G_l^t(B) \subset G_l^t(A) \cap B$. To prove the opposite inclusion, let $O_B$ be a neighbourhood of zero in $B$ and let $b \in G_l^t(A) \cap B$. Then there are a neighhbourhood $O_A$ of zero in $A$ such that $O_B = O_A \cap B$ and a neighbourhood $U_A$ of zero in $A$ such that $U_A b + U_A \subset O_A$. Moreover, there is a net $(a_\lambda)_{\lambda\in\Lambda}$ in $A$ such that $(a_\lambda b)_{\lambda\in\Lambda}$ converges in $A$ to $e_A$. Therefore, there is an index $\lambda_0 \in \Lambda$ such that $a_\lambda b - e_A \in U_A$ whenever $\lambda \succ \lambda_0$. Fix now an index $\lambda_1 \in \Lambda$ such that $\lambda_1 \succ \lambda_0$. Then $a_{\lambda_1} b - e_A \in U_A$. Since $B$ is dense in $A$, then there exists a net $(b_\alpha)_{\alpha\in\mathscr{A}}$ in $B$ which converges in $A$ to $a_{\lambda_1}$. Hence, there is an index $\alpha_0 \in \mathscr{A}$ such that $b_\alpha - a_{\lambda_1} \in U_A$ whenever $\alpha \succ \alpha_0$. Taking this into account,

$$b_\alpha b - e_A = (b_\alpha - a_{\lambda_1})b + (a_{\lambda_1} b - e_A) \in U_A b + U_A \subset O_A$$

whenever $\alpha \succ \alpha_0$. Hence, $(b_\alpha b)_{\alpha\in\mathscr{A}}$ converges to $e_A$ in $B$. It means that $b \in G_l^t(B)$.

The proof for right topological invertible elements is similar. □

**Corollary 5.** *Let $A$ be a unital left $TQ$-algebra* (*right $TQ$-algebra and $TQ$-algebra*) *and $B$ a dense subalgebra of $A$ with the same unit element, then $B$ is a left $TQ$-algebra* (*respectively, right $TQ$-algebra and $TQ$-algebra*).

**Proposition 5.** *Let $A$ be a unital left $TQ$-algebra* (*right $TQ$-algebra and $TQ$-algebra*) *and $I$ a closed two-sided ideal in $A$, then the quotient algebra $A/I$ is a left $TQ$-algebra* (*right $TQ$-algebra and $TQ$-algebra*).

*Proof.* Let $a \in G_l^t(A)$. Then there is a net $(a_\lambda)_{\lambda\in\Lambda}$ in $A$ such that $(a_\lambda a)_{\lambda\in\Lambda}$ converges in $A$ to $e_A$. Let $\pi : A \to A/I$ be the canonical map and $\tau_\pi$ the quotient topology on $A/I$ defined by $\pi$. Since $\pi$ is a continuous map, then $(\pi(a_\lambda)\pi(a))_{\lambda\in\Lambda}$ converges in $A/I$ to $e_{A/I} = \pi(e_A)$. It means that $\pi(G_l^t(A)) \subset G_l^t(A/I)$. Since $G_l^t(A)$ is open in $A$, $e_A \in G_l^t(A)$, and $\pi$ is an open map, then $\pi(G_l^t(A))$ is a neighbourhood of $e_{A/I}$ in $A/I$. Hence the interior part of $G_l^t(A/I)$ is not empty. Therefore $A/I$ (in the topology $\tau_\pi$) is a left $TQ$-algebra by Theorem 1.

The proof for right topological invertible elements is similar. □

## 4. TOPOLOGICAL IDEALS

Let $A$ be a unital semitopological algebra. We introduce the concept of a topological ideal and use it to characterize commutative complete metrizable unital $TQ$-algebras.

We say that a left (right) ideal $I$ in $A$ is a *topological left* (respectively, *right*) *ideal* if $I$ does not contain left (respectively, right) topologically invertible elements. We call a *topological ideal* an ideal which is a left topological ideal and a right topological ideal. Moreover, we call such ideal *maximal* if these are not contained in a larger topological ideal.

**Proposition 6.** *Every topological ideal in a semitopological unital algebra is contained in a maximal topological ideal.*

*Proof.* If $(I_\alpha)$ is a chain of left topological ideals (i.e. for two indices $\alpha \neq \beta$ we have either $I_\alpha \subset I_\beta$ or $I_\beta \subset I_\alpha$), then $\bigcup I_\alpha$ is also a left topological ideal and the conclusion follows from the Kuratowski–Zorn lemma.

The proofs for right and two-sided ideals are similar. □

**Proposition 7.** *All left* (*right*) *ideals in a unital semitopological algebra are topological if and only if $G_l^t(A) = G_l(A)$* (*respectively, $G_r^t(A) = G_r(A)$*).

*Proof.* Let $a \in G_l^t(A)$. If all left ideals in $A$ are topological, then none of the left ideals of $A$ can contain $a$. Hence, $a \in G_l(A)$. Therefore $G_l^t(A) = G_l(A)$. Conversely, if $G_l^t(A) = G_l(A)$, then every left ideal of $A$ is topological.

The proof for right ideals is similar. □

Atzmon in [13] constructed a complete locally convex commutative unital algebra in which all non-zero elements are topologically invertible and which is not a field. In this example the only maximal topological ideal is the zero ideal, while there are many dense maximal non-topological ideals.

**Proposition 8.** *Let $A$ be a unital semitopological algebra, and $M$ a closed maximal topological ideal*[13] *in $A$. Then all non-zero elements in the quotient algebra $A/M$ are topologically invertible.*

[13] Here and later on an ideal means a two-sided ideal.

*Proof.* If $J'$ is a closed topological ideal in $A/M$, then its inverse image $J$ under the quotient map is a (proper) closed ideal in $A$, and so it is a topological ideal. Since $M \subset J$, we have $J = M$, or $J'$ is a zero ideal in $A/M$. Thus all non-zero ideals in $A/M$ are dense and so all non-zero elements in $A/M$ are topologically invertible. The conclusion follows. □

**Proposition 9.** *Let $A$ be a unital left* (*right*) *$TQ$-algebra. Then all maximal topological left* (*respectively*, *right*) *ideals of $A$ are closed.*

*Proof.* Let $M$ be a maximal topological left ideal in $A$. Then $M \subset A \setminus G_l^t(A)$. Since $A$ is a left $TQ$-algebra, then $A \setminus G_l^t(A)$ is a closed subset in $A$. Therefore $\mathrm{cl}_A M \subset A \setminus G_l^t(A)$. Hence, $\mathrm{cl}_A M \cap G_l^t(A) = \emptyset$. It means that $\mathrm{cl}_A M$ is a topological left ideal as well, which implies $M = \mathrm{cl}_A M$. Hence, all maximal topological left ideals in $A$ are closed.

The proof for right ideals is similar. □

We shall prove now a topological version of the following result given in [7].

**Theorem A.** *Let $A$ be a commutative complete metrizable unital algebra. Then $A$ has all maximal ideals closed if and only if it is a $Q$-algebra.*

Our result reads as follows.

**Proposition 10.** *Let $A$ be a commutative complete metrizable unital algebra. Then $A$ has all maximal topological ideals closed if and only if it is a $TQ$-algebra.*

*Proof.* If $A$ is a $TQ$-algebra, then all maximal topological ideals are closed in $A$ by Proposition 9. If now $A$ is not a $TQ$-algebra, then, by Theorem 1, we can find a sequence $(x_i)$ of elements of $A$ which tends to $e_A$ and consists of elements which are not topologically invertible. By Lemma 2 in [33], we can find a subsequence $(a_i) \subset (x_i)$ such that all products $u_s = a_s a_{s+1} \ldots, s = 1, 2, \ldots$ are convergent and

$$\lim_s u_s = e_A. \tag{3}$$

Put $I_s = u_s A$. Since a product $xy$ is in $G^t(A)$ if and only if both $x$ and $y$ are in $G^{(t)}(A)$ (see [6], Lemma 3), all $I_s$ are topological ideals in $A$ and consequently $I = \bigcup I_s$ is also such an ideal. Since for every $x$ in $A$ and every natural $s$ the element $u_s x$ is in $I$, relation (3) implies that $I$ is dense in $A$. By Proposition 6, $A$ has a dense maximal topological ideal and the conclusion follows. □

**Remark 1.** The result of Proposition 10 can be void (equal to the Theorem A) since we know no example of a $TQ$-algebra of type $F$ that is not a $Q$-algebra. However, in [35] it is conjectured that the algebra constructed in [30] and similar algebras (called *Williamson type algebras*), which are $B_0$-algebras, have all non-zero elements topologically invertible, and so they are $TQ$-algebras of type $F$. Thus there is some hope that the result will be non-void.

**Remark 2.** Some results of the present paper have been recently independently obtained by also other authors (see [18], [22], and [26]).

## 5. OPEN PROBLEMS

We have several open problems connected with $TQ$-algebras.

**Problem 1.** *Does there exist a proper $TQ$-algebra of type $F$*?

**Problem 2.** *Does there exist an infinite dimensional $F$-algebra with all non-zero elements topologically invertible*?

**Problem 3.** *Does there exist a semitopological* (*or a topological*) *algebra with $G^t(A) \neq \mathfrak{G}^t(A)$*?

**Problem 4.** *Is the complexification of a real unital left* (*right*) *$TQ$-algebra a left* (*respectively*, *right*) *$TQ$-algebra*?

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## $TQ$-algebrate omadusi

Mati Abel ja Wiesław Żelazko

On vaadeldud ühe- ja kahepoolsete $TQ$-algebrate ning ühe- ja kahepoolsete topoloogiliste ideaalide põhiomadusi. On esitatud näiteid $TQ$-algebratest ja sõnastatud mõned senini lahendamata probleemid.