Proceedings of the Estonian Academy of Sciences,
2011, **60**, 2, 88–97
doi: 10.3176/proc.2011.2.03
Available online at www.eap.ee/proceedings

MATHEMATICS

# Approximation in variation by the Meyer-König and Zeller operators

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Received 29 January 2010, accepted 7 September 2010

**Abstract.** The convergence in variation and the rate of approximation of the Meyer-König and Zeller operators are discussed. It is proved that for absolutely continuous functions the rate of approximation can be estimated via the total variation.

**Key words:** approximation theory, Meyer-König and Zeller operators, functions of bounded variation, convergence in variation.

## 1. INTRODUCTION

In this paper the convergence in variation of the Meyer-König and Zeller operators is discussed. These operators have been investigated in many papers (see, for example, [1] and literature cited there). The operators of Meyer-König and Zeller [6] in the modification of Cheney and Sharma [5], defined by

$$\begin{aligned}(M_nf)(x) &= (1-x)^{n+1}\sum_{k=0}^{\infty}\binom{k+n}{k}x^k f\left(\frac{k}{k+n}\right) \quad (x\in[0,1)),\\ (M_nf)(1) &= f(1),\end{aligned} \tag{1.1}$$

are also called Bernstein power series.

Let $TV[0,1]$ denote the class of all functions of bounded variation on $[0,1]$, i.e. the total variation $V_{[0,1]}[f]$ of these functions is finite. We are interested in the convergence in variation of $M_n$, i.e. we study for $f\in TV[0,1]$ the quantity $V_{[0,1]}[M_nf-f]$. Since $M_nf\in AC[0,1]$, for every $f\in TV[0,1]$, as shown in [4], the convergence $V_{[0,1]}[M_nf-f]\to 0$ implies that $f$ has to be absolutely continuous; write $f\in AC[0,1]$.

Let $\Phi$ be the set of all real-valued strictly increasing convex functions $\varphi$ defined on $[0,1]$ such that $\varphi(0)=0$. For $\varphi\in\Phi$ and any complex-valued function $f$ defined on $[0,1]$, the $\varphi$-variation of $f$ on $[0,1]$ is defined by

$$V_\varphi(f;[0,1]) := \sup\sum_{i=1}^{n}\varphi(|f(x_i)-f(x_{i-1})|),$$

where the supremum is taken over all sequences $x_0\le x_1\le\ldots\le x_n$ with $x_0,x_n\in[0,1]$. In the particular case $\varphi(x)=x^p,\ p\ge 1$, the $\varphi$-variation is called $p$-variation and we write $V_p$ instead of $V_\varphi$. The notion of $\varphi$-variation, for any convex function $\varphi$, was introduced by Young [10] and extensively studied by Musielak and Orlicz in [7]. The general probabilistic approach to the convergence in $\varphi$-variation for linear positive operators can be found in [2], where conditions for the convergence in $\varphi$-variation via the usual modulus

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of continuity are given. However, it seems that this general approach does not apply to the convergence in ordinary variation. At least for the Bernstein operator it follows (see [2], Theorem 4 and Remark 4) that

$$V_p(B_n f - f;[0,1]) \le 2^{2p-1} V_{[0,1]}[f]\,\omega^{p-1}\left(f;\frac{1}{2\sqrt{n}}\right) \qquad (p>1)$$

for $f \in C_{[0,1]} \bigcap TV_{[0,1]}$, and here the case $p=1$ is meaningless.

We intend to find conditions for the convergence in ordinary variation via the variation of the function and the variation of the central differences of the function. For the convergence in variation it is important to state the variation detracting property. Let $L_n$ be linear positive operators acting on functions on $[0,1]$. If for all $f \in TV[0,1]$ we have $L_n f \in TV[0,1]$ and

$$V_{[0,1]}[L_n f] \le V_{[0,1]}[f], \tag{1.2}$$

then it is said that the operator $L_n$ has the variation detracting property.

The variation detracting property of linear positive operators was investigated in [2–4,8]. Adell and de la Cal (see [2], Theorem 1) gave the proof of the variation detracting property for some families of Bernstein-type operators based on the probabilistic approach. Here we present the direct proof of the same result in the case of the Meyer-König and Zeller operators. First we give a technical result.

**Lemma.** *Let $f \in TV[0,1]$. Then for $x \in [0,1)$*

$$(M_n f)'(x) = (n+1)(1-x)^n \sum_{k=0}^{\infty} x^k \binom{k+n+1}{k}\left[f\left(\frac{k+1}{k+n+1}\right) - f\left(\frac{k}{k+n}\right)\right]. \tag{1.3}$$

The next result is not new (cf. [2], Theorem 1 and Example K), however, for the completeness of the presentation we will give an elementary proof.

**Theorem A.** *If $f \in TV[0,1]$, then $M_n f \in AC[0,1]$ and*

$$V_{[0,1]}[M_n f] \le V_{[0,1]}[f].$$

*Proof.* Since $f \in TV[0,1]$, $(M_n f)'$ is bounded by Lemma. Consequently, $M_n f \in AC[0,1]$ and its total variation is

$$V_{[0,1]}[M_n f] = \int_0^1 |(M_n f)'(x)|dx.$$

So, using (1.3) and interchanging the order of summation and integration, we obtain

$$V_{[0,1]}[M_n f] \quad \le \quad (n+1)\sum_{k=0}^{\infty}\binom{k+n+1}{k}\left|f\left(\frac{k+1}{k+n+1}\right) - f\left(\frac{k}{k+n}\right)\right| \int_0^1 x^k(1-x)^n dx.$$

By definition of the beta-function we get

$$(n+1)\binom{k+n+1}{k}\int_0^1 x^k(1-x)^n dx = 1,$$

hence

$$V_{[0,1]}[M_n f] \le \sum_{k=0}^{\infty}\left|f\left(\frac{k+1}{k+n+1}\right) - f\left(\frac{k}{k+n}\right)\right| \le V_{[0,1]}[f]. \qquad \square$$

In Section 2 we derive the rate of approximation of $M_n f$ for $f^{(k)} \in AC[0,1]$, $k = 0,1,2$. For this purpose let us note another form of the derivative as

$$(M_n f)'(x) = \frac{(1-x)^n}{x} \sum_{k=0}^{\infty} \binom{k+n}{k} f\left(\frac{k}{k+n}\right) [k-(n+k+1)x]x^k \quad (x \in (0,1)). \tag{1.4}$$

In the proof of Theorem 1 we need the sum moments for the operators (1.1). Let us define for $r = 0,1,2,...$ (cf. (1.4)):

$$T_{r,n}(x) := (1-x)^{n+1} \sum_{k=0}^{\infty} [k-(n+k+1)x]^r \binom{k+n}{k} x^k. \tag{1.5}$$

Then by computations we have

$$T_{r+1,n}(x) = (n+1)x\Big(\sum_{l=0}^{r} \binom{r}{l} T_{l,n+1}(x) - T_{r,n}(x)\Big). \tag{1.6}$$

Since $T_{0,n}(x) \equiv 1$, by (1.6) we have (see (1.5))

$$T_{r,n}(x) = \begin{cases} 0, & r=1, \\ (n+1)x, & r=2, \\ (n+1)X, & r=3, \\ (n+1)x\,[1+3(n+2)x+X]\,, & r=4, \\ (n+1)X\,[(1+10(n+2)x+x^2], & r=5, \\ (n+1)x\,[1+15(n+2)X+15(n+2)^2x^2 & \\ +10(n+2)X+25(n+2)xX+(1+x^2)X], & r=6, \end{cases} \tag{1.7}$$

where $X := x(1+x)$.

## 2. CONVERGENCE IN VARIATION

In the following we study the rate of approximation of smooth functions with respect to the variation seminorm.

**Theorem 1.** *If* $g'' \in AC[0,1]$, *then*

$$V_{[0,1]}[M_n g - g] \le \frac{5\pi^3}{16n}(V_{[0,1]}[g] + V_{[0,1]}[g'']) \quad (n \ge 3). \tag{2.1}$$

*Proof.* We use for (1.4) Taylor's formula with the integral remainder term

$$g(t) = g(x) + (t-x)g'(x) + (t-x)^2\frac{g''(x)}{2} + \frac{1}{2}\int_0^{t-x}(t-x-u)^2 g'''(x+u)du.$$

Taking $t = \frac{k}{k+n}$, we have

$$(M_n g)'(x) = A_0(x)g(x) + A_1(x)g'(x) + A_2(x)(x)g''(x) + (R_n g)(x), \tag{2.2}$$

where

$$A_j(x) := \frac{(1-x)^n}{x} \sum_{k=0}^{\infty} [k-(n+k+1)x] \left(\frac{k}{k+n} - x\right)^j \binom{k+n}{k} x^k \quad (j = 0,1,2). \tag{2.3}$$

Calculating by (1.5) and (1.7), we get

$$A_0(x) = \frac{T_{1,n}(x)}{x(1-x)} = 0.$$

Analogously, using the property of the binomial coefficients, and moments $T_{2,n-1}(x)$ and $T_{1,n-1}(x)$, we obtain

$$\begin{aligned} A_1(x) &= \frac{(1-x)^n}{x}\sum_{k=0}^{\infty}([k-(n+k)x]-x)[k-(n+k)x]\frac{1}{k+n}\binom{k+n}{k}x^k \\ &= \frac{(1-x)^n}{nx}\sum_{k=0}^{\infty}[k-(n+k)x]^2\binom{k+n-1}{k}x^k - \frac{(1-x)^n}{n}\sum_{k=0}^{\infty}[k-(n+k)x]\binom{k+n-1}{k}x^k \\ &= \frac{1}{nx}T_{2,n-1}(x) - \frac{1}{n}T_{1,n-1}(x) = 1. \end{aligned}$$

We decompose the term $A_2(x)$ into two parts by summands of $1-\frac{1}{k+n}$, hence

$$\begin{aligned} A_2(x) &= \frac{(1-x)^n}{2x}\sum_{k=0}^{\infty}([k-(n+k-1)x]-2x)([k-(n+k-1)x]-x)^2 \\ &\quad \times\ x^k\frac{1}{n(n-1)}\binom{k+n-2}{k}\left(1-\frac{1}{k+n}\right) \\ &=: B_1 - B_2. \end{aligned} \tag{2.4}$$

Here, first removing the parentheses in the infinite sum and reordering by powers of $k-(n+k-1)$, and then using (1.5) and (1.7), we may write

$$\begin{aligned} B_1 &= \frac{1}{2n(n-1)}\frac{1-x}{x}[T_{3,n-2}(x) - 4xT_{2,n-2}(x) + 5x^2T_{1,n-2}(x) - 2x^3T_{0,n-2}(x)] \\ &= \frac{1}{2n}(1-x)(1-3x) - \frac{1}{n(n-1)}x^2(1-x). \end{aligned} \tag{2.5}$$

For $B_2$ we have

$$\begin{aligned} B_2 &:= B_2(n,x) := \frac{1}{2n(n-1)}\frac{(1-x)^n}{x}\sum_{k=0}^{\infty}\frac{1}{k+n}\Big([k-(n+k-1)x]-2x\Big) \\ &\quad \times\Big([k-(n+k-1)x]-x\Big)^2 x^k\binom{k+n-2}{k}, \end{aligned} \tag{2.6}$$

and after using Cauchy's inequality, the estimate $\frac{1}{k+n} < \frac{1}{n}$, and the definition of the moments in (1.5), we obtain

$$\begin{aligned}|B_2| &\leq \frac{1}{2n^2(n-1)}\frac{1-x}{x}\\ &\times\Big[(1-x)^{n-1}\sum_{k=0}^{\infty}([k-(n+k-1)x]-2x)^2\binom{k+n-2}{k}x^k\Big]^{\frac{1}{2}}\\ &\times\Big[(1-x)^{n-1}\sum_{k=0}^{\infty}([k-(n+k-1)x]-x)^4\binom{k+n-2}{k}x^k\Big]^{\frac{1}{2}}\\ &= \frac{1}{2n^2(n-1)}\frac{1-x}{x}\Big[T_{2,n-2}(x)-4xT_{1,n-2}(x)+4x^2T_{0,n-2}(x)\Big]^{\frac{1}{2}}\\ &\times\Big[T_{4,n-2}(x)-4xT_{3,n-2}(x)+6x^2T_{2,n-2}(x)-4x^3T_{1,n-2}(x)+x^4T_{0,n-2}(x)\Big]^{\frac{1}{2}}. \qquad (2.7)\end{aligned}$$

Now, by (2.4) and (2.5), we have

$$A_2(x)=\frac{1}{2n}(1-x)(1-3x)-\frac{1}{n(n-1)}x^2(1-x)-B_2(n,x). \qquad (2.8)$$

From (2.7), using (1.7), we get

$$|B_2(n,x)|\leq\frac{2}{n\sqrt{n}}, \qquad n\geq 3.$$

By (2.2) and (2.8) we may estimate

$$\begin{aligned}\int_0^1\Big|(M_ng)'(x)-g'(x)\Big|dx &\leq \frac{1}{2n}\int_0^1(1-x)|1-3x||g''(x)|dx\\ &+\frac{1}{n(n-1)}\int_0^1 x^2(1-x)|g''(x)|dx+\frac{2}{n\sqrt{n}}\int_0^1|g''(x)|dx+\|R_ng\|\\ &\leq \Big(\frac{1}{2n}+\frac{4}{27n(n-1)}+\frac{2}{n\sqrt{n}}\Big)\|g''\|+\|R_ng\|, \qquad (2.9)\end{aligned}$$

where the norm is taken in $L^1(0,1)$, i.e. $\|f\|:=\|f\|_{L^1(0,1)}$.

The integral remainder term in (2.2) has by (1.4) the form

$$\begin{aligned}(R_ng)(x) &:= \frac{(1-x)^n}{2x}\sum_{k=0}^{\infty}\frac{1}{(k+n)^2}[k-(n+k+1)x]\binom{k+n}{k}x^k\\ &\times\int_0^{\frac{k}{k+n}-x}[k-(x+u)(k+n)]^2g'''(x+u)du.\end{aligned}$$

We denote $R_ng=:R_n^1g-R_n^2g$, where

$$\begin{aligned}(R_n^1g)(x) &:= \frac{(1-x)^n}{2x}\sum_{k=0}^{\infty}\frac{1}{(k+n)^2}[k-(n+k)x]\binom{k+n}{k}x^k\\ &\times\int_0^{\frac{k}{k+n}-x}[k-(x+u)(k+n)]^2g'''(x+u)du\end{aligned}$$

and

$$(R_n^2g)(x) := \frac{(1-x)^n}{2}\sum_{k=0}^{\infty}\frac{1}{(k+n)^2}\binom{k+n}{k}x^k\int_0^{\frac{k}{k+n}-x}[k-(x+u)(k+n)]^2g'''(x+u)du.$$

In integrals of the previous quantities $(R_n^1 g),(R_n^2 g)$ we substitute $x+u=v$ and note that $|\frac{k}{k+n}-v|^2 \leq |\frac{k}{k+n}-x|^2$ for $\frac{k}{k+n} \leq v \leq x$ or $x \leq v \leq \frac{k}{k+n}$. As $k-(n+k)x$ and the integral $\int_x^{\frac{k}{k+n}}|...|dv$ have the same sign, we have that

$$\begin{aligned} ||R_n^1 g|| &\leq \frac{1}{2}\int_0^1 \frac{(1-x)^n}{x}\sum_{k=0}^{\infty}\frac{1}{(k+n)^2}[k-(n+k)x]\binom{k+n}{k}x^k \\ &\quad \times \int_x^{\frac{k}{k+n}}[k-v(k+n)]^2|g'''(v)|dvdx \\ &\leq \frac{1}{2}\int_0^1 \frac{(1-x)^n}{x}\sum_{k=0}^{\infty}\frac{1}{(k+n)^2}[k-(n+k)x]^3\binom{k+n}{k}x^k \\ &\quad \times \left(\int_0^{\frac{k}{k+n}}-\int_0^x\right)|g'''(v)|dvdx =: S_n^2 g - S_n^1 g, \end{aligned} \tag{2.10}$$

where

$$S_n^1 g := \frac{1}{2}\int_0^1 \frac{(1-x)^n}{x}\sum_{k=0}^{\infty}\frac{1}{(k+n)^2}[k-(n+k)x]^3\binom{k+n}{k}x^k\int_0^x|g'''(v)|dvdx \tag{2.11}$$

and

$$S_n^2 g := \frac{1}{2}\int_0^1 \frac{(1-x)^n}{x}\sum_{k=0}^{\infty}\frac{1}{(k+n)^2}[k-(n+k)x]^3\binom{k+n}{k}x^k\int_0^{\frac{k}{k+n}}|g'''(v)|dvdx. \tag{2.12}$$

To estimate $S_n^1 g$ in (2.11), let

$$U_n^1 := \frac{(1-x)^n}{x}\sum_{k=0}^{\infty}\frac{1}{(k+n)^2}[k-(n+k)x]^3\binom{k+n}{k}x^k. \tag{2.13}$$

Then, after decomposing the previous sum, we obtain

$$U_n^1 = \frac{1}{n}\frac{(1-x)^n}{x}\sum_{k=0}^{\infty}\left(\frac{1}{k+n-1}-\frac{1}{(k+n)(k+n-1)}\right)[k-(n+k)x]^3\binom{k+n-1}{k}x^k.$$

Using (1.5) and (1.7), we have

$$\begin{aligned} U_n^1 &= \frac{(1-x)^n}{n(n-1)x}\sum_{k=0}^{\infty}([k-(n+k-1)x]-x)^3\binom{k+n-2}{k}x^k - U_n^{1,1} \\ &= \frac{1}{n(n-1)}\frac{(1-x)}{x}[T_{3,n-2}(x)-3xT_{2,n-2}(x)+3x^2T_{1,n-2}(x)-x^3]-U_n^{1,1} \\ &= \frac{1}{n(n-1)}\frac{(1-x)}{x}[(n-1)x(1-2x)-x^3]-U_n^{1,1}, \end{aligned} \tag{2.14}$$

where

$$U_n^{1,1} := \frac{1}{n(n-1)}\frac{(1-x)^n}{x}\sum_{k=0}^{\infty}\frac{1}{k+n}([k-(n+k-1)x]-x)^3\binom{k+n-2}{k}x^k.$$

To estimate $U_n^{1,1}$, we use Cauchy's inequality

$$|U_n^{1,1}| \leq \frac{1}{n^2(n-1)}\frac{1-x}{x}\Big[(1-x)^{n-1}\sum_{k=0}^{\infty}([k-(n+k-1)x]-x)^4\binom{k+n-2}{k}x^k\Big]^{\frac{1}{2}}$$
$$\times\Big[(1-x)^{n-1}\sum_{k=0}^{\infty}([k-(n+k-1)x]-x)^2\binom{k+n-2}{k}x^k\Big]^{\frac{1}{2}},$$

and using (1.5), we have

$$|U_n^{1,1}| \leq \frac{1}{n^2(n-1)}\frac{1-x}{x}\Big[(T_{4,n-2}(x)-4xT_{3,n-2}(x)+6x^2T_{2,n-2}(x)-4x^3T_{1,n-2}(x)+x^4)\Big]^{\frac{1}{2}}$$
$$\times\Big[(T_{2,n-2}(x)-2xT_{1,n-2}(x)+x^2)\Big]^{\frac{1}{2}}.$$

Application of (1.7) gives ($n \geq 3$)

$$\begin{aligned} |U_n^{1,1}| &\leq \frac{1}{n^2(n-1)}(1-x)\Big[3(n-1)^2x+(n-1)(1+3x^2)+x^3\Big]^{\frac{1}{2}}[n-1+x]^{\frac{1}{2}} \\ &= \frac{1}{n^2}(1-x)\Big[3x+\frac{1+3x^2}{n-1}+\frac{x^3}{(n-1)^2}\Big]^{\frac{1}{2}}\sqrt{n-1}\Big[1+\frac{x}{n-1}\Big]^{\frac{1}{2}} \\ &\leq \frac{1}{n\sqrt{n}}(1-x)\Big[3x+\frac{1+3x^2}{2}+\frac{x^3}{4}\Big]^{\frac{1}{2}}\Big[1+\frac{x}{2}\Big]^{\frac{1}{2}} \leq \frac{3}{n\sqrt{n}}. \end{aligned}$$

Therefore, using (2.14), we have

$$|U_n^1| \leq \frac{1}{n}+\frac{4}{27n(n-1)}+\frac{3}{n\sqrt{n}}.$$

Taking into account the connection between $S_n^1 g$ in (2.11) and $U_n^1$ in (2.13), we have

$$|S_n^1 g| \leq \Big(\frac{1}{2n}+\frac{2}{27n(n-1)}+\frac{3}{2n\sqrt{n}}\Big)\int_0^1 |g'''(v)|dv. \quad (2.15)$$

Applying Fubini's theorem to $S_n^2 g$ in (2.12), we have

$$|S_n^2 g| \leq \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(k+n)^2}\Big|\int_0^{\frac{k}{k+n}}|g'''(v)|dv\binom{k+n}{k}\int_0^1[k-(n+k)x]^3(1-x)^n x^{k-1}dx\Big|.$$

The inner integral can be evaluated by using the beta-function and then estimated by

$$\Big|\binom{k+n}{k}\int_0^1[k-(n+k)x]^3(1-x)^n x^{k-1}dx\Big| = \Big|\frac{5k^2n+3kn^2-2n^3+6k^2}{(n+k+1)(n+k+2)(n+k+3)}\Big| \leq 2.$$

Now

$$\begin{aligned} |S_n^2 g| &\leq \sum_{k=0}^{\infty}\frac{1}{(k+n)^2}\int_0^{\frac{k}{k+n}}|g'''(v)|dv \leq \|g'''\|\sum_{i=0}^{\infty}\sum_{k=in}^{(i+1)n-1}\frac{1}{(k+n)^2} \\ &\leq \frac{\|g'''\|}{n}\sum_{i=0}^{\infty}\frac{1}{(i+1)^2} = \frac{\|g'''\|}{n}\frac{\pi^2}{6}. \end{aligned} \quad (2.16)$$

Returning back to $||R_n^1 g||$ (cf. (2.10)), we have, using (2.16) and (2.15),

$$||R_n^1 g|| \leq \frac{\pi^2||g'''||}{6n} + \frac{||g'''||}{2n} + \frac{2||g'''||}{27n(n-1)} + \frac{3||g'''||}{2n\sqrt{n}}. \tag{2.17}$$

Analogously to $R_n^1 g$ in (2.10), we have

$$\begin{aligned} ||R_n^2 g|| &\leq \frac{1}{2}\int_0^1 (1-x)^n \sum_{k=0}^{\infty} \frac{1}{(k+n)^2}[k-(n+k)x]^2 \binom{k+n}{k} x^k \\ &\quad \times \int_x^{\frac{k}{k+n}} |g'''(v)|dvdx =: S_n^4 g - S_n^3 g, \end{aligned} \tag{2.18}$$

where

$$S_n^3 g := \frac{1}{2}\int_0^1 (1-x)^n \sum_{k=0}^{\infty} \frac{1}{(k+n)^2}[k-(n+k)x]^2 \binom{k+n}{k} x^k \int_0^x |g'''(v)|dvdx \tag{2.19}$$

and

$$S_n^4 g := \frac{1}{2}\int_0^1 (1-x)^n \sum_{k=0}^{\infty} \frac{1}{(k+n)^2}[k-(n+k)x]^2 \binom{k+n}{k} x^k \int_0^{\frac{k}{k+n}} |g'''(v)|dvdx. \tag{2.20}$$

To estimate $S_n^3 g$, let (cf. (2.19))

$$U_n^2 := (1-x)^n \sum_{k=0}^{\infty} \frac{1}{(k+n)^2}[k-(n+k)x]^2 \binom{k+n}{k} x^k. \tag{2.21}$$

Then, decomposing the sum, using (1.5) and (1.7), we have

$$\begin{aligned} U_n^2 &= \frac{1}{n}(1-x)^n \sum_{k=0}^{\infty} \left(\frac{1}{k+n-1} - \frac{1}{(k+n)(k+n-1)}\right)[k-(n+k)x]^2 \binom{k+n-1}{k} x^k \\ &= \frac{1}{n(n-1)}(1-x)^n \sum_{k=0}^{\infty} ([k-(n+k-1)x]-x)^2 \binom{k+n-2}{k} x^k - U_n^{2,1} \\ &= \frac{1}{n(n-1)}(1-x)[T_{2,n-2}(x) - 2xT_{1,n-2}(x) + x^2] - U_n^{2,1} \\ &= \frac{1}{n(n-1)}(1-x)[(n-1)x + x^2] - U_n^{2,1}, \end{aligned} \tag{2.22}$$

where

$$U_n^{2,1} := \frac{1}{n(n-1)}(1-x)^n \sum_{k=0}^{\infty} \frac{1}{k+n}([k-(n+k-1)x]-x)^2 \binom{k+n-2}{k} x^k.$$

Again, by $\frac{1}{k+n} \leq \frac{1}{n}$, using (1.5) and (1.7), we obtain

$$U_n^{2,1} \leq \frac{1}{n^2(n-1)}(1-x)[(n-1)x + x^2]. \tag{2.23}$$

Using (2.22) and (2.23), we have

$$|S_n^3 g| \leq \left(\frac{1}{8n} + \frac{2}{27n(n-1)} + \frac{1}{8n^2} + \frac{2}{27n^2(n-1)}\right)\int_0^1 |g'''(v)|dv. \tag{2.24}$$

Applying Fubini's theorem to $S_n^4 g$ in (2.20), we have

$$|S_n^4 g| \leq \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{(k+n)^2}\Big|\int_0^{\frac{k}{k+n}}|g'''(v)|dv\binom{k+n}{k}\int_0^1[k-(n+k)x]^2(1-x)^n x^k dx\Big|.$$

The inner integral can be evaluated by using the beta-function which gives the estimate

$$\Big|\binom{k+n}{k}\int_0^1[k-(n+k)x]^2(1-x)^n x^k dx\Big| = \Big|\frac{k^2n+kn^2+2k^2-2kn+2n^2}{(n+k+1)(n+k+2)(n+k+3)}\Big| \leq \frac{1}{3}.$$

Thus, for $S_n^4 g$ we get by (2.16)

$$|S_n^4 g| \leq \frac{1}{6}\sum_{k=0}^{\infty}\frac{1}{(k+n)^2}\|g'''(v)\| \leq \frac{\pi^2}{36n}\|g'''\|.$$

By (2.18), collecting the estimates of $S_n^4 g$ and $S_n^3 g$ in (2.24), we have

$$||R_n^2 g|| \leq \frac{\pi^2||g'''||}{36n} + \Big(\frac{1}{8n}+\frac{2}{27n(n-1)}+\frac{1}{8n^2}+\frac{2}{27n^2(n-1)}\Big)||g'''||. \tag{2.25}$$

Finally, since $R_n g = R_n^1 g - R_n^2 g$, we obtain by (2.9) the estimate (see (2.2), (2.17), and (2.25))

$$\begin{aligned} ||(M_n g)' - g'|| \leq\ & \frac{||g''||}{2n} + \frac{4||g''||}{27n(n-1)} + \frac{2||g''||}{n\sqrt{n}} + \frac{\pi^2||g'''||}{6n} \\ & + \frac{||g'''||}{2n} + \frac{2||g'''||}{27n(n-1)} + \frac{3||g'''||}{2n\sqrt{n}} + \frac{\pi^2||g'''||}{36n} \\ & + \Big(\frac{1}{8n}+\frac{2}{27n(n-1)}+\frac{1}{8n^2}+\frac{2}{27n^2(n-1)}\Big)||g'''||. \end{aligned}$$

Now we use Stein's inequality with the exact constant (see, e.g., [9], Theorem A10.1)

$$||g''||_{L^1} \leq \frac{\pi^3}{16}\sqrt{||g'||_{L^1}||g'''||_{L^1}}$$

and the inequality for the geometric and arithmetic means. We have

$$||(M_n g)' - g'|| \leq \frac{5\pi^3}{16n}(||g'|| + ||g'''||), \quad (n \geq 3),$$

which finishes our proof. □

Below $(\overline{\triangle}_h^r g)(x)$ denotes the central difference of $g$ of order $r$,

$$(\overline{\triangle}_h^r g)(x) := \sum_{k=0}^{r}(-1)^k\binom{r}{k}g\left(x+\left(\frac{r}{2}-k\right)h\right).$$

The proof of the next theorem is essentially presented in [4], Theorem 3.7.

**Theorem B.** *Let the linear positive operator $L_n$ satisfy the variation detracting property* (1.2) *and let it satisfy for any $g'' \in AC[0,1]$ the condition*

$$V_{[0,1]}[L_n g - g] \leq \frac{C}{n}(V_{[0,1]}[g] + V_{[0,1]}[g'']). \tag{2.26}$$

*If $f \in AC[0,1]$, then there exist constants $c_1, c_2 > 0$ such that*

$$V_{[0,1]}[L_n f - f] \le c_1 \sup_{0<h\le n^{-\frac{1}{2}}} V_{[h,1-h]}[\overline{\triangle}_h^2 f] + \frac{c_2}{n} V_{[0,1]}[f].$$

*In particular, if $f' \in AC[0,1]$, then*

$$V_{[0,1]}[L_n f - f] \le \frac{c_1}{\sqrt{n}} \sup_{0<h\le n^{-\frac{1}{2}}} V_{[\frac{h}{2},1-\frac{h}{2}]}[\overline{\triangle}_h^1 f'] + \frac{c_2}{n} V_{[0,1]}[f].$$

Since by Theorem A the operator $M_n$ has the variation detracting property (1.2) and by Theorem 1 it satisfies the condition (2.26), we finally have for the Meyer-König and Zeller operators

**Theorem 2.** *If $f \in AC[0,1]$, then there exist constants $c_1, c_2 > 0$ such that*

$$V_{[0,1]}[M_n f - f] \le c_1 \sup_{0<h\le n^{-\frac{1}{2}}} V_{[h,1-h]}[\overline{\triangle}_h^2 f] + \frac{c_2}{n} V_{[0,1]}[f].$$

*In particular, if $f' \in AC[0,1]$, then*

$$V_{[0,1]}[M_n f - f] \le \frac{c_1}{\sqrt{n}} \sup_{0<h\le n^{-\frac{1}{2}}} V_{[\frac{h}{2},1-\frac{h}{2}]}[\overline{\triangle}_h^1 f'] + \frac{c_2}{n} V_{[0,1]}[f].$$

## ACKNOWLEDGEMENTS

This research was partially supported by the Estonian Science Foundation (grants 7033 and 8627) and by the Estonian Ministry of Education and Research, project SF0132723s06.

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## Meyer-Königi ja Zelleri operaatoritega lähendamine variatsiooni mõttes

Andi Kivinukk ja Tarmo Metsmägi

On uuritud Meyer-Königi ja Zelleri operaatoritega lähendamise kiirust, mida mõõdetakse funktsiooni (või selle tuletiste) täisvariatsiooniga. On tõestatud, et kui lähendatav funktsioon on absoluutselt pidev (või vastavalt esimene ehk teine tuletis on absoluutselt pidevad), siis lähendamise kiirus on hinnatav funktsiooni (vastavalt selle tuletiste) täisvariatsiooni kaudu.