Proceedings of the Estonian Academy of Sciences,
2011, **60**, 2, 81–87
doi: 10.3176/proc.2011.2.02
Available online at www.eap.ee/proceedings

MATHEMATICS

On determinability of idempotent medial commutative quasigroups by their endomorphism semigroups

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Received 8 April 2010, accepted 17 June 2010

Abstract. We extend the result of P. Puusemp (Idempotents of the endomorphism semigroups of groups. *Acta Comment. Univ. Tartuensis*, 1975, 366, 76–104) about determinability of finite Abelian groups by their endomorphism semigroups to finite idempotent medial commutative quasigroups.

Key words: idempotent medial commutative quasigroup, Abelian group, endomorphism, commutative Moufang loop.

1. INTRODUCTION

In this paper we study the endomorphism semigroups of idempotent medial commutative quasigroups (IMC-quasigroups, for short). K. Toyoda established a connection between medial quasigroups and Abelian groups (Theorem 2.10 in [2]). Endomorphism rings of Abelian groups have been studied by several authors and the obtained results are presented in [4]. In [6] Puusemp proved that if G and G' are finite Abelian groups, then the isomorphism $G \cong G'$ follows from the isomorphism between their endomorphism semigroups $\operatorname{End} G \cong \operatorname{End} G'$ (more precisely, it was proved that if G is a group such that its endomorphism semigroup is isomorphic to the endomorphism semigroup of a finite Abelian group, then the groups G and G' are isomorphic). Motivated by Toyoda's result (Theorem 2.10 in [2]) and Puusemp's result, we study endomorphisms of magmas (groupoids) which are "very close" to Abelian groups. To be more precise, we replace the associativity by a weaker assumption – mediality. It is known that every Abelian group G has the zero-endomorphism corresponding to the maximal congruence $G \times G$. There exist medial quasigroups with no proper subquasigroups, such that all their endomorphisms are invertible. Motivated by results of endomorphism semigroups of groups, we restrict ourselves to the medial quasigroups with zero-endomorphisms. For this purpose we consider idempotent quasigroups.

As a result we generalize Puusemp's result to finite IMC-quasigroups, that is, if the endomorphism semigroups of finite IMC-quasigroups Q and Q' are isomorphic, then the quasigroups Q and Q' are isomorphic too.

Idempotent medial commutative quasigroups arise in several examples of mid-point quasigroups. Let $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real numbers and the set of positive reals, respectively. Define two binary operations $x \boxplus y = (x+y)/2$ and $x \odot y = \sqrt{xy}$. Then both $(\mathbb{R}, \boxplus)$ and $(\mathbb{R}_+, \odot)$ are IMC-quasigroups.

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The paper is organized as follows. In Section 2 we present the necessary definitions and propositions needed for the main theorem. These results are elementary and can be found also in [3,5]. For the convenience of the reader, we recall them together with the proofs.

The connection between the given ICM-quasigroup and associated commutative Moufang loops will be studied in Section 3. The main theorem will be given and proved in Section 4.

2. CONNECTION BETWEEN IMC-QUASIGROUPS AND ABELIAN GROUPS

Let us start by recalling the classical definition of the quasigroup.

Definition 1. *A magma $\langle Q,\cdot\rangle$ is called a quasigroup if each of the equations $ax=b$ and $ya=b$ has a unique solution for any $a,b\in Q$.*

The solutions of these equations will be denoted by $x=a\backslash b$ and $y=b/a$, respectively. We also need the following definition of quasigroups.

Definition 2. *A set Q with three binary operations $\cdot$, $\backslash, /$ is called a quasigroup if the following identities hold*:

$$\begin{aligned} x\backslash(x\cdot y) &= x\cdot(x\backslash y)=y,\\ (y\cdot x)/x &= (y/x)\cdot x=y. \end{aligned}$$

Definitions 1 and 2 are equivalent (see [2]). Next it is assumed that Q is a quasigroup $\langle Q,\cdot,\backslash,/\rangle$.

It follows from the definitions that the mappings $L_a,R_b\colon Q\to Q$, defined by $L_a(x)=ax$, $R_b(x)=xb$, are bijective. Hence, to each quasigroup Q one can associate the subgroup $M(Q)=\langle\{L_a,R_b\mid a,b\in Q\},\circ\rangle$ of the group of all bijections $Q\to Q$. The group $M(Q)$ is called a multiplication group or an associated group of the quasigroup Q.

The mapping $\varphi\colon Q\to Q$ is called an endomorphism of Q if φ preserves the binary operation $\cdot$, that is $\varphi(x\cdot y)=\varphi(x)\cdot\varphi(y)$ for all $x,y\in Q$. An invertible endomorphism of Q is called an automorphism of Q. The set of all endomorphisms (automorphisms) of Q will be denoted by End Q (resp. Aut Q). By abuse of notation, we let End Q stand for the endomorphism monoid of Q. Immediate computations show that if $\varphi\in$ End Q, then φ preserves also the binary operations $\backslash$ and $/$.

Definition 3. *A quasigroup Q is called medial* (*commutative*) *if it satisfies the identity $(x\cdot y)\cdot(u\cdot v)=(x\cdot u)\cdot(y\cdot v)$* (*resp. $x\cdot y=y\cdot x$*).

Similarly to Abelian groups, all endomorphisms of a medial quasigroup Q are summable, i.e. if φ,ψ are endomorphisms of a medial quasigroup Q, then $\varphi+\psi$ defined by $(\varphi+\psi)(x)=\varphi(x)\cdot\psi(x)$ is an endomorphism too.

Theorem 1 (Toyoda's theorem (Theorem 2.10 in [2]))**.** *If Q is a medial quasigroup*, *then there exist an Abelian group $\langle Q,+,-,0\rangle$*, *its commuting automorphisms ϕ and ψ*, *and an element $c\in Q$ such that*

$$x\cdot y=\phi(x)+\psi(y)+c. \tag{1}$$

The Abelian group $\langle Q,+,-,0\rangle$ is called an underlying Abelian group of the medial quasigroup Q.

Definition 4. *A quasigroup Q is called idempotent if in Q the identity $x\cdot x=x$ is fulfilled.*

Proposition 1. *If Q is an idempotent medial quasigroup*, *then*
1. *Q is a distributive quasigroup*, *that is*, *Q satisfies both $x\cdot(y\cdot z)=(x\cdot y)\cdot(x\cdot z)$ and $(y\cdot z)\cdot x=(y\cdot x)\cdot(z\cdot x)$*;
2. *$M(Q)$ is a subgroup of* Aut *Q.*

The proof is straightforward.

If $e \in Q$ is an idempotent, then $\langle \{e\}, \cdot \rangle$ is a subquasigroup in Q. From now on, we write π_e for the endomorphism $Q \to \{e\}$. Clearly, π_e is a left-zero in End Q, i.e. $\pi_e \circ \varphi = \pi_e$ for each $\varphi \in$ End Q.

Proposition 2. *If Q is an idempotent quasigroup, then φ is a left-zero in* End Q *iff $\varphi = \pi_e$ for some $e \in Q$.*

Proof. If $\varphi \in$ End Q, then obviously $\pi_e \circ \varphi = \pi_e$ for each $e \in Q$.

Conversely, suppose that an endomorphism θ is left-zero in End Q. For each $e, x \in Q$ we have

$$\theta(x) = (\theta \circ \pi_e)(x) = \theta(e) = \pi_{\theta(e)}(x).$$

Hence, $\theta = \pi_f$, where $f = \theta(e)$. □

Proposition 3. *Suppose that $\langle Q, \cdot \rangle$ is a medial quasigroup, where $\cdot$ is in the form* (1). *Then the following hold*:
1. *Q is commutative iff $\phi = \psi$*;
2. *Q is idempotent iff $c = 0$ and $\phi + \psi = 1_Q$.*

See also Theorem 4 in [7].

Let $\mathbb{2}$ denote the endomorphism $x \mapsto x + x$ of an Abelian group $\langle Q, +, -, 0 \rangle$.

Proposition 4. *A quasigroup Q is an IMC-quasigroup iff there exists an Abelian group $\langle Q, +, -, 0 \rangle$ such that the mapping $\mathbb{2}$ is its automorphism and $x \cdot y = \mathbb{2}^{-1}(x + y)$.*

Proof. Suppose that $\langle Q, \cdot \rangle$ is an IMC-quasigroup. It follows from Toyoda's theorem that $x \cdot y = \phi(x) + \psi(y) + c$, where ϕ and ψ are automorphisms of $\langle Q, +, -, 0 \rangle$. Since $\langle Q, \cdot \rangle$ is commutative and idempotent, Proposition 3 shows that $\phi = \psi$, $c = 0$, and $1_Q = \phi + \phi = \mathbb{2} \circ \phi$. As ϕ is an automorphism, we have that $\mathbb{2}^{-1} = \phi$ and finally that $x \cdot y = \mathbb{2}^{-1}(x + y)$.

Conversely, if $x \cdot y = \mathbb{2}^{-1}(x + y)$, then it is staightforward to check that $\langle Q, \cdot \rangle$ is an IMC-quasigroup. □

One should note that any IMC-quasigroup is uniquely determined by its underlying Abelian group. Next we assume that everywhere $\langle Q, + \rangle$ is an underlying Abelian group of the given IMC-quasigroup $\langle Q, \cdot \rangle$.

Corollary 1. *If the Abelian group $\langle Q, +, -, 0 \rangle$ is finite, then $\mathbb{2}$ is an automorphism iff $\langle Q, +, -, 0 \rangle$ is of odd order.*

The next corollary is a special case of Proposition 3 in [8].

Corollary 2. *If Q is an IMC-quasigroup, then* $\mathrm{End}(Q, +) \hookrightarrow \mathrm{End}(Q, \cdot)$ (*embedding of monoids*).

Indeed, if η is an endomorphism of the Abelian group $\langle Q, +, -, 0 \rangle$, then $\mathbb{2} \circ \eta = \eta \circ \mathbb{2}$, $\mathbb{2}^{-1} \circ \eta = \eta \circ \mathbb{2}^{-1}$ and for each $x, y \in Q$ we have

$$\begin{aligned} \eta(x) \cdot \eta(y) &= \mathbb{2}^{-1}(\eta(x) + \eta(y)) = \mathbb{2}^{-1}(\eta(x)) + \mathbb{2}^{-1}(\eta(y)) \\ &= \eta(\mathbb{2}^{-1}(x + y)) = \eta(x \cdot y), \end{aligned}$$

i.e. $\eta \in \mathrm{End}(Q, \cdot)$.

Corollary 3. *If Q is an IMC-quasigroup, then $x \backslash y = y / x = \mathbb{2}(y) - x$.*

Indeed, since $y = (y/x) \cdot x$, we have $y = \mathbb{2}^{-1}(y/x + x)$, i.e. $y/x = \mathbb{2}(y) - x$. In a commutative quasigroup always $y/x = x \backslash y$. So we have $x \backslash y = \mathbb{2}(y) - x$.

Definition 5. *Two quasigroups $\langle Q, \cdot \rangle$ and $\langle Q', * \rangle$ are called isomorphic, if there exist the bijective mapping $\tau \colon Q \to Q'$, such that $\tau(x \cdot y) = \tau(x) * \tau(y)$ for each $x, y \in Q$.*

3. CONNECTION BETWEEN IMC-QUASIGROUPS AND COMMUTATIVE MOUFANG LOOPS

It was shown in Proposition 1 that any idempotent medial quasigroup is distributive. Therefore, any IMC-quasigroup is also a distributive quasigroup. Distributive quasigroups are connected to commutative Moufang loops (see [2] for more details).

Definition 6. *A quasigroup $\langle Q, \cdot \rangle$ is called a loop if there exists $e \in Q$ such that $e \cdot x = x = x \cdot e$ for each $x \in Q$.*

We denote it by $\langle Q, \cdot, e \rangle$.

Definition 7. *A loop $\langle Q, \cdot, e \rangle$ is called a commutative Moufang loop if it satisfies the identity*

$$(x \cdot x) \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z). \tag{2}$$

From Definition 7 it follows that the binary operation $\cdot$ is commutative and there exists the mapping ${}^{-1}\colon Q \to Q$ such that $x^{-1} \cdot (x \cdot y) = y$ holds.

Let $\langle Q, \cdot \rangle$ be an IMC-quasigroup and $k \in Q$. Let us define a new binary operation on Q as follows:

$$x \oplus_k y = R_k^{-1} x \cdot L_k^{-1} y. \tag{3}$$

The magma $\langle Q, \oplus_k \rangle$ is a commutative Moufang loop with the identity element k by Theorem 8.1 in [2]. Due to commutativity of $\langle Q, \cdot \rangle$, we have that

$$x \oplus_k y = L_k^{-1} x \cdot L_k^{-1} y = L_k^{-1}(x \cdot y).$$

Moreover, the unary operation ${}^{-1}$ of $\langle Q, \oplus_k \rangle$ will be denoted by $\ominus_k$ and the element $\ominus_k x$ coincides with k/x.

Proposition 5. *The commutative Moufang loop $\langle Q, \oplus_k, \ominus_k, k \rangle$ is an Abelian group.*

Proof (see also Theorem 9 in [5]). It is sufficient to prove only the associativity of $\oplus_k$. Let $x, y, z \in Q$. Then

$$\begin{aligned}
x \oplus_k (y \oplus_k z) &= (x \oplus_k k) \oplus_k (y \oplus_k z) = L_k^{-1}((x \oplus_k k) \cdot (y \oplus_k z)) \\
&= L_k^{-1}\left(L_k^{-1}(x \cdot k) \cdot L_k^{-1}(y \cdot z)\right) = L_k^{-1}\left(L_k^{-1}((x \cdot k) \cdot (y \cdot z))\right) \\
&= L_k^{-1}\left(L_k^{-1}((x \cdot y) \cdot (k \cdot z))\right) = L_k^{-1}\left(L_k^{-1}(x \cdot y) \cdot L_k^{-1}(k \cdot z)\right) \\
&= L_k^{-1}((x \oplus_k y) \cdot (k \oplus_k z)) = (x \oplus_k y) \oplus_k (k \oplus_k z) \\
&= (x \oplus_k y) \oplus_k z. \qquad \square
\end{aligned}$$

Corollary 4. *If $\langle Q, \cdot \rangle$ is an IMC-quasigroup with the underlying Abelian group $\langle Q, +, -, 0 \rangle$, that is $x \cdot y = \mathfrak{2}^{-1}(x + y)$, then $x \oplus_k y = x + y - k$.*

Proof. Let $x, y \in Q$. Then

$$\begin{aligned}
x \oplus_k y &= L_k^{-1}(x \cdot y) = k \backslash (x \cdot y) = \mathfrak{2}(x \cdot y) - k = \mathfrak{2}\left(\mathfrak{2}^{-1}(x + y)\right) - k \\
&= x + y - k. \qquad \square
\end{aligned}$$

As a particular case we have

Corollary 5. *Let the assumptions be as in Corollary 4. Then $x \oplus_0 y = x + y$.*

By the last corollary we have

$$x + y = x \oplus_0 y = L_0^{-1}(x \cdot y) = L_0^{-1}(\mathcal{2}^{-1}(x+y)).$$

Therefore, $\mathcal{2} = L_0^{-1}$, i.e. L_0^{-1} is an automorphism of the underlying Abelian group. We have a more general result.

Corollary 6. *Let $\mathcal{2}_k$, with $\mathcal{2}_k(x) = x \oplus_k x$, be an automorphism of the Abelian group $\langle Q, \oplus_k, \ominus_k, k\rangle$. Then $\mathcal{2}_k = L_k^{-1}$.*

Obviously,

$$\mathcal{2}_k(x) = x \oplus_k x = L_k^{-1}(x \cdot x) = L_k^{-1}(x).$$

Proposition 6. *For each $k, l \in Q$ the Abelian groups $\langle Q, \oplus_k, \ominus_k, k\rangle$ and $\langle Q, \oplus_l, \ominus_l, l\rangle$ are isomorphic.*

Proof (see also Theorem 10 in [5]). For each $k \in Q$, the mapping $\psi_k \colon Q \to Q$, given by $\psi_k(x) = x + k$, is an isomorphism from $\langle Q, +, -, 0\rangle$ to $\langle Q, \oplus_k, \ominus_k, k\rangle$. Now the proposition follows immediately. □

We will write 0^Q for the set of all left-zero endomorphisms of an ICM-quasigroup Q. By Proposition 2,

$$0^Q = \{\pi_k \colon Q \to \{k\} \mid k \in Q\}.$$

Let $k \in Q$. The set of all submonoids M of $\mathrm{End}(Q, \cdot)$ such that $1_Q, \pi_k \in M$ and $M \cap 0^Q = \{\pi_k\}$ is non-empty and, by Zorn's lemma, has a maximal element. From now on, M_k denotes a maximal submonoid in $\mathrm{End}(Q, \cdot)$ such that $M_k \cap 0^Q = \{\pi_k\}$.

Proposition 7. *The submonoid M_k coincides with the endomorphism monoid of the Abelian group $\langle Q, \oplus_k, \ominus_k, k\rangle$.*

Proof. The proof is divided into three steps:
1. $\varphi(k) = k$ for each $\varphi \in M_k$;
2. $\varphi(x \oplus_k y) = \varphi(x) \oplus_k \varphi(y)$ for each $\varphi \in M_k$ and for each $x, y \in Q$;
3. the endomorphism monoid $\mathrm{End}(Q, \oplus_k)$ of the Abelian group $\langle Q, \oplus_k, \ominus_k, k\rangle$ coincides with M_k.

It follows from the first two steps that $M_k \subseteq \mathrm{End}(Q, \oplus_k)$.

1. Let $\varphi \in M_k$. By definition, $\pi_k \in M_k$. Therefore $\varphi \circ \pi_k \in M_k$. For each $x \in Q$ we have $(\varphi \circ \pi_k)(x) = \varphi(k)$ i.e. $\varphi \circ \pi_k = \pi_{\varphi(k)}$. By the definition of M_k we conclude that $\pi_k = \pi_{\varphi(k)}$ and hence $k = \varphi(k)$.
2. Let $\varphi \in M_k$ and let $x, y \in Q$. Then

$$\begin{aligned} \varphi(x \oplus_k y) &= \varphi(L_k^{-1}(x \cdot y)) = \varphi(k \backslash (x \cdot y)) = \varphi(k) \backslash (\varphi(x) \cdot \varphi(y)) \\ &= L_{\varphi(k)}^{-1}(\varphi(x) \cdot \varphi(y)) = L_k^{-1}(\varphi(x) \cdot \varphi(y)) \\ &= \varphi(x) \oplus_k \varphi(y). \end{aligned}$$

3. By Corollary 6, $\mathcal{2}_k = L_k^{-1}$. Hence, for each $\xi \in \mathrm{End}(Q, \oplus_k)$ we have $\mathcal{2}_k \circ \xi = \xi \circ \mathcal{2}_k$, $L_k^{-1} \circ \xi = \xi \circ L_k^{-1}$, $\xi \circ L_k = L_k \circ \xi$ and, in view of $x \oplus_k y = L_k^{-1}(x \cdot y)$, it follows that

$$\xi(x \cdot y) = \xi(L_k(x \oplus_k y)) = L_k(\xi(x) \oplus_k \xi(y)) = \xi(x) \cdot \xi(y),$$

i.e. $\xi \in \mathrm{End}(Q, \cdot)$. Therefore, $M_k \subseteq \mathrm{End}(Q, \oplus_k) \subseteq \mathrm{End}(Q, \cdot)$. On the other hand, it is easy to check that $\mathrm{End}(Q, \oplus_k) \cap 0^Q = \{\pi_k\}$ by the definition of M_k, it follows that $\mathrm{End}(Q, \oplus_k) = M_k$.
The proposition is proved. □

Corollary 7. *Let $\varphi \in \mathrm{End}(Q, \cdot)$. Then $\varphi \in M_k \Leftrightarrow \varphi(k) = k$.*

Proof. Let

$$M' = \{\varphi \in \mathrm{End}(Q,\cdot) \mid \varphi(k) = k\}.$$

Obviously $1_Q \in M'$ and $\varphi \circ \psi \in M'$ whenever $\varphi, \psi \in M'$. Hence, M' is a monoid and $\pi_k \in M'$ by the definition of π_k. The first part of the proof of Proposition 7 implies $M_k \subseteq M'$. Clearly, M' is a submonoid of $\mathrm{End}(Q,\cdot)$ such that $M' \cap 0^Q = \{\pi_k\}$. By the definition of M_k we have $M' = M_k$. □

Corollary 8. *If* $\mathrm{End}(Q,\cdot)$ *is finite, then* $(Q,\cdot)$ *is also finite.*

We give two proofs for this corollary. The first one is more elementary. The second proof uses results of the group theory and the analogue of its corollary for groups.

Elementary proof. If $\mathrm{End}(Q,\cdot)$ is finite, then 0^Q is finite too. Hence the IMC-quasigroup Q is finite due to the one-to-one correspondence between 0^Q and Q, i.e. $k \leftrightarrow \pi_k$. □

Group-theoretic proof. Let $\mathrm{End}(Q,\cdot)$ be finite. Since $\mathrm{End}(Q,+) \subseteq \mathrm{End}(Q,\cdot)$, the monoid $\mathrm{End}(Q,+)$ is finite, too. It is well known that if the endomorphism monoid of a group G is finite, then the group G is finite by Theorem 2 in [1]. Therefore, the group $(Q,+)$ is finite and so is the IMC-quasigroup $(Q,\cdot)$. □

4. MAIN THEOREM

Theorem 2. *Let* $\langle Q,\cdot\rangle$ *and* $\langle Q',*\rangle$ *be IMC-quasigroups and* $\langle Q,\cdot\rangle$ *be finite. If the endomorphism monoids* $\mathrm{End}(Q,\cdot)$ *and* $\mathrm{End}(Q',*)$ *are isomorphic, then the quasigroups Q and Q' are isomorphic.*

Proof. By Corollary 8, Q' is finite, too. Let $x, y \in Q$ and $x', y' \in Q'$. By Proposition 4 we have

$$x \cdot y = \mathcal{2}^{-1}(x+y) \quad \text{and} \quad x' * y' = \mathcal{2}'^{-1}(x' +' y'),$$

where $\langle Q,+\rangle$ and $\langle Q',+'\rangle$ are the underlying Abelian groups of Q and Q', respectively. By Corollary 2, the monoids $\mathrm{End}(Q,+)$ and $\mathrm{End}(Q',+')$ are contained in the monoids $\mathrm{End}(Q,\cdot)$ and $\mathrm{End}(Q',*)$, respectively.

Let $M \leqslant \mathrm{End}(Q,\cdot)$ be the maximal submonoid, such that $M \cap 0^Q = \{\pi_k\}$ for some $k \in Q$. By Propositions 6 and 7 and Corollary 5 we have an isomorphism $M \cong \mathrm{End}(Q,+)$. Let $\Gamma\colon \mathrm{End}(Q,\cdot) \to \mathrm{End}(Q',*)$ be an isomorphism of monoids. Hence, the image of the restriction of Γ to $\mathrm{End}(Q,+)$ is the maximal submonoid $M' \leqslant \mathrm{End}(Q',*)$ such that $M' \cap 0^{Q'} = \{\pi_{k'}\}$ for some $k' \in Q'$. From Propositions 6 and 7 we conclude that $M' \cong \mathrm{End}(Q',+')$ and finally that $\mathrm{End}(Q,+) \cong \mathrm{End}(Q',+')$.

Since $\langle Q,+\rangle$ and $\langle Q',+'\rangle$ are finite Abelian groups, their isomorphism follows from $\mathrm{End}(Q,+) \cong \mathrm{End}(Q',+')$ (see Theorem 4.2 in [6]). Let $\xi\colon Q \to Q'$ be the corresponding isomorphism. Clearly, $\mathcal{2}' \circ \xi = \xi \circ \mathcal{2}$ and $\mathcal{2}'^{-1} \circ \xi = \xi \circ \mathcal{2}^{-1}$. We finish the proof by showing that ξ is also an isomorphism between the quasigroups Q and Q'. Let $x, y \in Q$. Then

$$\begin{aligned}
\xi(x) * \xi(y) &= \mathcal{2}'^{-1}(\xi(x) +' \xi(y)) = \mathcal{2}'^{-1}(\xi(x)) +' \mathcal{2}'^{-1}(\xi(y)) \\
&= \xi(\mathcal{2}^{-1}(x)) +' \xi(\mathcal{2}^{-1}(y)) = \xi(\mathcal{2}^{-1}(x) + \mathcal{2}^{-1}(y)) \\
&= \xi(\mathcal{2}^{-1}(x+y)) \\
&= \xi(x \cdot y).
\end{aligned}$$

□

Corollary 9. *Let endomorphism monoids of two IMC-quasigroups Q and Q' be isomorphic. If Q is finite, then the underlying Abelian groups of Q and Q' are isomorphic.*

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Idempotentsete mediaalsete kommutatiivsete kvaasirühmade määratavusest oma endomorfismipoolrühmadega

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On uuritud lõplike kommutatiivsete idempotentsete mediaalsete kvaasirühmade määratavust oma endomorfismipoolrühmaga. Lähtudes Toyoda teoreemist, mis seob omavahel mediaalsed kvaasirühmad ja Abeli rühmad, ning lõplike Abeli rühmade määratavusest oma endomorfismipoolrühmaga [6], on mainitud tulemust laiendatud lõplikele kommutatiivsetele idempotentsetele mediaalsetele kvaasirühmadele. On näidatud, et ka selliste kvaasirühmade jaoks saab laiendada rühmateooriast tuntud tulemust, et rühma G endomorfismipoolrühma lõplikkusest järeldub rühma lõplikkus (järeldus 8).