Proceedings of the Estonian Academy of Sciences,
2011, **60**, 2, 69–80
doi: 10.3176/proc.2011.2.01
Available online at www.eap.ee/proceedings

MATHEMATICS

Ford lemma for topological *-algebras

Mart Abel and Mati Abel*

Institute of Pure Mathematics, University of Tartu, J. Liivi 2, 50409 Tartu, Estonia

Received 9 March 2010, accepted 12 October 2010

Abstract. Several analogies of Ford lemma for topological algebras (in particular, for topological *-algebras) are proved (without using projective limits). Topological *-algebras, in which a self-adjoint element a with $\mathrm{sp}_A(a) \subset (0,\infty)$ has a self-adjoint square root b with $\mathrm{sp}_A(a) \subset (0,\infty)$ and $\mathrm{sp}_A(h_1 + \ldots + h_n) \subset [0,\infty)$, if $\mathrm{sp}_A(h_k) \subset [0,\infty)$ where h_k are self-adjoint elements for each $k \in \{1,\ldots,n\}$, are described.

Key words: advertive algebras, Ford lemma, Gelfand–Mazur algebras, locally pseudoconvex algebras, simplicial algebras, square root of an element, topological *-algebras.

At several places in the development of the theory of topological *-algebras (especially, of the theory of Banach *-algebras and locally m-convex *-algebras) a self-adjoint square root for a self-adjoint element with a positive spectrum is needed.

In 1966, James W. M. Ford proved in his doctoral dissertation [20] (see also [21]; [13], Proposition 8.13 and 12.11; [33], Theorem 3.4.5; [34], Proposition 11.1.7; [41], Lemmas 9.8, 9.10 and Corollary 9.9) a general square root lemma for Banach algebras and Banach *-algebras. This result was generalized for complete locally m-convex *-algebras in [26], Lemma 1 and Corollary (see also [38], Theorems 3.9 and 3.10; [22], Theorems 5.5.4, 5.5.8 and Corollary 5.5.5; and [15], Proposition 1.12 and Corollary 1.13); for p-Banach *-algebras in [18], Proposition 3.1; for pseudocomplete locally convex *-algebras in [35], Lemma 1; for complete locally m-convex *-algebras with not necessarily bounded spectrum in [39], Theorem 2.2; for complete locally m-pseudoconvex Hausdorff *-algebras in [11], Proposition 5.3.4 and Corollary 5.3.5 (for several kinds of locally pseudoconvex algebras see [14], Proposition 5.1; [15], Proposition 4.6; [16], Proposition 2.4; and [17], Proposition 3.1), and for fundamental Fréchet algebras in [10], Theorems 3.2 and 3.3.

In the present paper all these results are generalized (without using projective limits) to the case of topological algebras and topological *-algebras with continuous involution or not necessarily continuous involution.

1. INTRODUCTION

Let A be a topological algebra over the field of complex numbers $\mathbb{C}$ with separately continuous multiplication (in short, a topological algebra), $\hom A$ the set of all nontrivial characters of A, and $m(A)$ the set of all closed

* Corresponding author, mati.abel@ut.ee

regular (or modular) two-sided ideals in A which are maximal as left ideals or as right ideals. Then A/M (in the quotient topology) is a division Hausdorff algebra for all $M \in m(A)$ (see [23], Theorem 24.9.6, and [33], Theorem 2.4.12). Here A/M could be topologically isomorphic to $\mathbb{C}$ or not. It is known that there exist topological division algebras which are not topologically isomorphic to $\mathbb{C}$ (see, for example, [42], pp. 83 and 85, or [40], pp. 731 and 732). When A/M is topologically isomorphic to $\mathbb{C}$ for each $M \in m(A)$, then A is called a *Gelfand–Mazur algebra*. Hence, if $m(A)$ is empty, then A is always a Gelfand–Mazur algebra, but when $m(A)$ is nonempty, then most of topological algebras are Gelfand–Mazur algebras.

It is well known (see, for example, [1], Lemma 1.11; [2], Corollary 2; or [8], Theorem 3.3) that all p-normed algebras with $p \in (0,1]$, all locally m-convex algebras, all locally convex Fréchet algebras, all locally m-pseudoconvex algebras and many more general topological algebras are Gelfand–Mazur algebras. Indeed, Gelfand–Mazur algebras are exactly the class of topological algebra for which the Gelfand theory, well known in the case of commutative Banach algebras, works. If A is a Gelfand–Mazur algebra and $m(A)$ is not empty, then every $M \in m(A)$ has the form $M = \ker \varphi$ for some $\varphi \in \hom A$. In this case every commutative Gelfand–Mazur algebra A is homomorphic/isomorphic with a subalgebra of $C(\hom A)$, similarly to the case of commutative Banach algebras.

A topological algebra is *locally pseudoconvex* (*locally m-pseudoconvex*) if it has a base of neighbourhoods of zero consisting of balanced and pseudoconvex[1] (respectively, balanced, idempotent[2], and pseudoconvex) sets. It is well known that the topology of a locally pseudoconvex (locally m-pseudoconvex) algebra can be given by a family of nonhomogeneous (respectively, nonhomogeneous and submultiplicative) seminorms[3]. In the particular case when the power of homogeneity $k \in (0,1]$ does not depend on the seminorms of this family, one speaks about locally k-convex and locally m-(k-convex) algebras and when $k = 1$, then about locally convex and locally m-convex algebras. It is well known that all locally convex and all locally bounded algebras[4] are locally pseudoconvex algebras and all locally m-convex algebras and all p-normed algebras with $p \in (0,1]$ are locally m-pseudoconvex algebras.

A topological algebra A is a *simplicial algebra* or a *normal algebra* (in the sense of Michael) if every closed regular two-sided ideal of A is contained in some closed maximal regular two-sided ideal of A. It is known that all commutative locally m-pseudoconvex (in particular, commutative locally m-convex) algebras are simplicial (see [5], Corollary 5; for the case of complete algebras see [4], Proposition 2, and [11], Corollary 7.1.14; and for the case of locally m-convex algebras see [42], p. 110, or [12], pp. 321 and 322).

An element a of a topological algebra A is called *topologically quasi-invertible* in A if there exist nets $(a_\lambda)_{\lambda\in\Lambda}$ and $(b_\mu)_{\mu\in M}$ in A such that $(a_\lambda \circ a)_{\lambda\in\Lambda}$ and $(a \circ b_\mu)_{\mu\in M}$ converge to the zero element θ_A of A (here $a \circ b = a + b - ab$ for every $a, b \in A$) and an element a of a unital topological algebra A is called *topologically invertible* in A if there exist nets $(a_\lambda)_{\lambda\in\Lambda}$ and $(b_\mu)_{\mu\in M}$ in A such that $(a_\lambda a)_{\lambda\in\Lambda}$ and $(ab_\mu)_{\mu\in M}$ converge to the unit element e_A of A.

Let Tqinv$A$ denote the set of all topologically quasi-invertible elements in A, Qinv$A$ the set of all quasi-invertible elements in A and, for a unital topological algebra A, let Tinv$A$ denote the set of all topologically invertible elements in A and Inv$A$ the set of all invertible elements in A. A topological algebra A is called an *advertive* topological algebra if Tqinv$A$ = Qinv$A$ and an *invertive* topological algebra if Tinv$A$ = Inv$A$. It is known (see [3], Proposition 2 and Corollary 2, or [28], p. 73) that all Q-algebras (that is, topological algebras in which Qinv$A$, in the unital case Inv$A$, is open) and all complete locally m-pseudoconvex algebras are advertive (in the unital case invertive).

Let A be a topological algebra. A Cauchy sequence (a_n) is called a *Mackey–Cauchy sequence* in A if there exist a balanced and bounded subset B of A and for every $\varepsilon > 0$ a number $n_\varepsilon \in \mathbb{N}$ such that $a_{n+m} - a_m \in \varepsilon B$ whenever $n > n_\varepsilon$ and $m > 0$. A topological algebra A is *sequentially Mackey complete*

---

[1] A set U in A is *pseudoconvex* if there is a $\mu > 0$ such that $U + U \subset \mu U$.

[2] A set U in A is *idempotent* if $UU \subset U$.

[3] A seminorm p on A is *nonhomogeneous* if $p(\lambda a) = |\lambda|^k p(a)$ for each $a \in A$ and $\lambda \in \mathbb{C}$, where the *power of homogeneity* is $k = k(p) \in (0,1]$, and p is *submultiplicative* if $p(ab) \leqslant p(a)p(b)$ for each $a, b \in A$.

[4] A topological algebra is *locally bounded* if it has a bounded neighbourhood of zero.

if every Mackey–Cauchy sequence of A converges in A. Hence, all complete topological algebras are sequentially Mackey complete (because every Mackey–Cauchy sequence[5] is a Cauchy sequence).

Let A be a topological *-algebra, that is, a topological algebra on which an involution $a \to a^*$ has been given. An element $a \in A$ is *self-adjoint* or *hermitian* if $a^* = a$.

Let again A be a topological algebra. If A has the unit element e_A, then

$$\sigma_A(a) = \{\lambda \in \mathbb{C} : a - \lambda e_A \notin \mathrm{Inv}A\},$$

and if A is an algebra without unit, then

$$\sigma_A(a) = \{\lambda \in \mathbb{C} \setminus \{0\} : \frac{a}{\lambda} \notin \mathrm{Qinv}A\} \cup \{0\}$$

is the (algebraic) *spectrum* of $a \in A$. In both cases

$$\rho_A^t(a) = \sup\{|\lambda| : \lambda \in \sigma_A^t(a)\}$$

is the (algebraic) *spectral radius* of A.

For noninvertive algebras with the unit element

$$\sigma_A^t(a) = \{\lambda \in \mathbb{C} : a - \lambda e_A \notin \mathrm{Tinv}A\}$$

and for nonadvertive algebras

$$\sigma_A^t(a) = \{\lambda \in \mathbb{C} \setminus \{0\} : \frac{a}{\lambda} \notin \mathrm{Tqinv}A\} \cup \{0\}$$

is the *topological spectrum* of $a \in A$. In both cases

$$\rho_A^t(a) = \sup\{|\lambda| : \lambda \in \sigma_A(a)\}$$

is the *topological spectral radius* of A.

Herewith, we take $\rho_A^t(a) = 0$ if $\sigma_A^t(a) = \emptyset$, and $\rho_A^t(a) = \infty$ if $\sigma_A^t(a)$ is an unbounded set in $\mathbb{C}$, similarly as in the case of the algebraic spectrum. It is easy to see that $\sigma_A^t(a) \subseteq \sigma_A(a)$ and $\rho_A^t(a) \leqslant \rho_A(a)$ for each $a \in A$.

Moreover, A is an advertive algebra if and only if $\sigma_A^t(a) = \sigma_A(a)$ for each $a \notin \mathrm{Qinv}A$. Indeed, if A is an advertive algebra, then $\sigma_A^t(a) = \sigma_A(a)$ for each $a \in A$. Let now $a \in A \setminus \mathrm{Qinv}A$. Then $1 \in \sigma_A(a)$. If now $\sigma_A(a) = \sigma_A^t(a)$, then $a \notin \mathrm{Tqinv}A$. Hence, $\mathrm{Qinv}A = \mathrm{Tqinv}A$ in this case. Therefore, A is an advertive algebra. Similarly, a unital topological algebra is an invertive algebra if and only if $\sigma_A(a) = \sigma_A^t(a)$ for each $a \notin \mathrm{Inv}A$ (see [9], p. 258).

Let now A be a topological *-algebra. Then

$$\sigma_A(a^*) = \{\overline{\mu} : \mu \in \sigma_A(a)\}$$

for each $a \in A$. Therefore, $\sigma_A(a) \subset \mathbb{R}$, similarly $\sigma_A^t(a) \subset \mathbb{R}$, if $a \in A$ is self-adjoint.

Let A be a topological algebra. Then

$$\beta_A(a) = \inf\left\{\lambda > 0 : \left\{\left(\frac{a}{\lambda}\right)^n : n \in \mathbb{N}\right\} \text{ is bounded in } A\right\}$$

is the *radius of boundedness* of $a \in A$. It satisfies the following conditions:

$$\beta_A(\mu a) = |\mu|\beta_A(a) \text{ and } \beta_A(a^k) = \beta_A(a)^k$$

---

[5] It is known (see [24], p. 122) that there exist Cauchy sequences which are not Mackey–Cauchy sequences.

for each $a \in A$, $\mu \in \mathbb{C}$, and $k \in \mathbb{N}$. If $a, b \in A$ and the product of any two idempotent bounded subsets of A is bounded, then

$$\beta_A(ab) \leq \beta_A(a)\beta_A(b),$$

and if, in addition, the convex hull of an idempotent and bounded set of A is bounded (in particular, A is a locally convex algebra with continuous multiplication), then

$$\beta_A(a+b) \leqslant \beta_A(a) + \beta_A(b)$$

(see [31], p. 281; [32], p. 310; and [19], Lemma II.9).

Herewith, if $\beta_A(a) < \infty$, then $a \in A$ is called a *bounded element* of A, and if all elements in A are bounded, then A is called a *topological algebra with bounded elements*.

2. FORD LEMMA FOR TOPOLOGICAL ALGEBRAS

Let A be an algebra and $a \in A$. An element $b \in A$ is called the *quasi-square root* of a if $b \circ b = a$. When A is a unital algebra and $b^2 = a$, then b is called the *square root* of a. For each $a \in A$ we put $S'(a) = \{(a)^n : n \geqslant 1\}$ and[6] $S(a) = \Gamma(S'(a))$.

First, we prove the following generalization of a result of Powell[7] (see [35], Lemma 1).

Theorem 2.1. *Let A be a sequentially Mackey complete topological algebra. If $a \in A$ and $S(a)$ is bounded in A, then there exists an element $b \in A$ such that $b \circ b = a$ and $\beta_A(b) \leqslant 1$. In particular, when $\beta_A(a) < 1$ and*

(a) $\rho_A(x) \leqslant \beta_A(x)$ *for each* $x \in A$

and

(b) $\beta_A(x+y) \leqslant \beta_A(x) + \beta_A(y)$ *if x and y commute in A,*

hold, then there is only one quasi-square root b of a such that $\beta_A(b) < 1$.

Proof. Let $a \in A$ be such that $S(a)$ is bounded in A. Then $S(a)$ is an idempotent and bounded subset of A. Since the closure $B = \mathrm{cl}(S(a))$ is a closed, idempotent, bounded and absolutely convex subset in A (see, for example, [27], pp. 103, and [30], pp. 5–6), then the subalgebra A_B of A, generated by B, is a normed algebra with respect to the submultiplicative norm $\|\cdot\|$, defined by

$$\|x\| = \inf\{|\lambda| : x \in \lambda B\}$$

for each $x \in A_B$, and the norm topology on A_B is not weaker than the topology on A_B induced by the topology of A (see [6], Proposition 2.2). Moreover, A_B is complete, because A is sequentially Mackey complete, and $\|a\| \leqslant 1$, because $a \in S(a) \subset B$.

For each $n \in \mathbb{N}$ we put

$$S_n = -\sum_{k=1}^{n} \binom{\frac{1}{2}}{k} (-a)^k.$$

Since

$$\| S_{n+l} - S_n \| \leqslant \left| \binom{\frac{1}{2}}{n+1} \right| \| a \|^{n+1} + \cdots + \left| \binom{\frac{1}{2}}{n+l} \right| \| a \|^{n+l} \leqslant \sum_{k=n+1}^{n+l} \left| \binom{\frac{1}{2}}{k} \right|$$

6 The *absolutely convex hull* of $S \subset A$ is the set

$$\Gamma(S) = \Big\{ \sum_{k=1}^{n} \lambda_k s_k : n \in \mathbb{N}, s_1, \ldots, s_n \in S \text{ and } \lambda_1, \ldots, \lambda_n \in \mathbb{C} \text{ with } \sum_{k=1}^{n} |\lambda_k| \leqslant 1 \Big\}.$$

7 He considered the case when A is a pseudocomplete locally convex *-algebra.

and the series

$$\sum_{k=1}^{\infty}\left|\binom{\frac{1}{2}}{k}\right|$$

converges (see [33], p. 361), it follows that (S_n) is a Cauchy sequence in A_B. Hence, (S_n) converges in A_B to

$$b = -\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}(-a)^k = \lim_{n\to\infty} S_n \in A_B \subset A$$

and

$$b\circ b = -2\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}(-a)^k - \sum_{k=1}^{\infty}\left[\sum_{s=1}^{k}\binom{\frac{1}{2}}{s}\binom{\frac{1}{2}}{k+1-s}\right](-a)^{k+1}.$$

Since

$$\sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta}{n-k} = \binom{\alpha+\beta}{n}$$

for each $\alpha,\beta\in\mathbb{R}$ and $n\in\mathbb{N}$ (see [36], formula 13, p. 616), then

$$\sum_{s=1}^{k}\binom{\frac{1}{2}}{s}\binom{\frac{1}{2}}{k+1-s} = -2\binom{\frac{1}{2}}{k+1}.$$

Hence

$$b\circ b = 2\binom{\frac{1}{2}}{1}a = a.$$

Since

$$\beta_A(x) \leqslant \beta_{A_B}(x) = \rho_{A_B}(x) = \| x \|$$

for each $x\in A_B$ and the norm is continuous on A_B, then

$$\beta_A(b) \leqslant \| b \| = \lim_{n\to\infty}\left\| -\sum_{k=1}^{n}\binom{\frac{1}{2}}{k}(-a)^k\right\| \leqslant \lim_{n\to\infty}\sum_{k=1}^{n}\left|\binom{\frac{1}{2}}{k}\right| \| a \|^k$$

$$= -\sum_{k=1}^{n}\binom{\frac{1}{2}}{k}(-\| a \|)^k = 1-\sqrt{1-\| a \|} \leqslant 1.$$

Let now $\beta_A(a)<1$, A satisfy the conditions (a) and (b), and let $c\in A$ be any element such that $c\circ c = a$ and $\beta_A(c)<1$. Then $c\circ c = a = b\circ b$, $2c = a+c^2$ and $2b = a+b^2$. Therefore,

$$2(b\circ c) = (a+b^2)+(a+c^2)-2bc = 2a+(b-c)^2 = 2a+(c-b)^2 = 2(c\circ b).$$

Hence, $cb = bc$. Taking this into account,

$$\rho_A\left(\frac{b+c}{2}\right) \leqslant \beta_A\left(\frac{b+c}{2}\right) \leqslant \frac{\beta_A(b)+\beta_A(c)}{2} < 1$$

by conditions (a) and (b). It means that

$$d = \frac{b+c}{2} \in \mathrm{Qinv}A.$$

Hence, there exists the quasi-inverse $e\in A$ for d. Therefore, from

$$b-c = \theta_A\circ(b-c) = (e\circ d)\circ(b-c) = e\circ\frac{b+c+2b-2c-(b^2-c^2)}{2}$$

$$= e\circ\frac{b+c+b\circ b-c\circ c}{2} = e\circ d = \theta_A$$

it follows that $b=c$. □

Corollary 2.2. *Let A be a sequentially complete locally pseudoconvex Hausdorff algebra. If $a \in A$ and*[8] *$S'(a)$ is bounded in A, then there exists an element $b \in A$ such that $b \circ b = a$ and $\beta_A(b) \leqslant 1$. In particular, when $\beta_A(a) < 1$, then there is only one quasi-square root b of a such that $\beta_A(b) < 1$.*

Proof. In the present case, A is sequentially Mackey complete and $\rho_A(a) \leqslant \beta_A(a)$ for each $a \in A$ (see [6], Corollary 4.3). Moreover, the convex hull of any idempotent and bounded set is bounded in A, because A is locally pseudoconvex and the convex hull of a set U is a subset of $\Gamma(U)$. Hence A satisfies condition (b) of Theorem 2.1 (see [31], p. 281). Consequently, Corollary 2.2 holds by Theorem 2.1. □

Corollary 2.3. *Let A be a sequentially complete locally m-convex Hausdorff algebra. If $a \in A$ and $\beta_A(a) < 1$, then in A there exists only one quasi-square root b of a such that $\beta_A(b) < 1$.*

Proof. In the present case A satisfies the conditions (a) and (b) of Theorem 2.1 (see [6], Proposition 4.1, and [19], Lemma II.9). Therefore, Corollary 2.2 completes the proof. □

For topological unital algebras we have

Corollary 2.4. *Let A be a unital sequentially Mackey complete topological algebra. If $a \in A$ and $S(e_A - a)$ ($S(e_A - \frac{a}{M})$ for some $M > 1$) is bounded in A, then there exists an element $b \in A$ such that $b^2 = a$ and $\beta_A(e_A - b) \leqslant 1$ (respectively, $\beta_A(e_A - \frac{b}{\sqrt{M}}) \leqslant 1$). In particular, when $\beta_A(e_A - a) < 1$ ($\beta_A(e_A - \frac{a}{M}) < 1$ for some $M > 1$) and A satisfies* (a) *and* (b) *of Theorem* 2.1*, then there is only one square root b for a with $\beta_A(e_A - b) < 1$ (respectively, $\beta_A(e_A - \frac{b}{\sqrt{M}}) < 1$).*

Proof. Since $S(e_A - a)$ is bounded in A, then there exists an element $c \in A$ such that $c \circ c = e_A - a$ or $(e_A - c)^2 = a$ and $\beta_A(c) \leqslant 1$. Hence $b = e_A - c$ is a square root of a and $\beta_A(e_A - b) \leqslant 1$. If now $\beta_A(e_A - a) < 1$ and A satisfies (a) and (b) of Theorem 2.1, then there is only one square root for a by Theorem 2.1.

If $S(e_A - \frac{a}{M})$ is bounded in A for some $M > 1$, then the proof is similar. □

Similarly to Corollaries 2.2 and 2.3 the following corollaries hold.

Corollary 2.5. *Let A be a unital sequentially complete locally pseudoconvex Hausdorff algebra. If $a \in A$ and $S'(e_A - a)$ is bounded in A, then there exists an element $b \in A$ such that $b^2 = a$ and $\beta_A(e_A - b) \leqslant 1$. In particular, when $\beta_A(e_A - a) < 1$, then there is only one square root b of a such that $\beta_A(e_A - b) < 1$.*

Corollary 2.6. *Let A be a unital sequentially complete locally m-convex Hausdorff algebra. If $a \in A$ and $\beta_A(e_A - a) < 1$, then in A there exists only one square root b of a such that $\beta_A(e_A - b) < 1$.*

Now we consider the case when A is a topological *-algebra.

Theorem 2.7. *Let A be a sequentially Mackey complete topological *-algebra. If $a \in A$ and $S(a)$ is bounded in A, then there exists an element $b \in A$ such that $b \circ b = a$ and $\beta_A(b) \leqslant 1$. In particular, when*

(c) *the involution $a \to a^*$ in A is continuous*

or

(d) *a has only one quasi-square root in A,*

then b is self-adjoint if a is self-adjoint.

Proof. By hypothesis and Theorem 2.1, there exists an element

$$b = -\sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} (-a)^k \in A$$

[8] Since A is locally pseudoconvex, then $S(a)$ is bounded by the boundedness of $S'(a)$.

such that $b \circ b = a$ and $\beta_A(b) \leqslant 1$. Let now $a^* = a$. If A satisfies (c), then

$$b^* = -\lim_{n\to\infty}\sum_{k=1}^{n}\binom{\frac{1}{2}}{k}(-a^*)^k = -\lim_{n\to\infty}\sum_{k=1}^{n}\binom{\frac{1}{2}}{k}(-a)^k = b$$

and if A satisfies condition (d), then $b^* = b$, because $b^* \circ b^* = a^* = a$. □

Corollary 2.8. *Let A be a unital sequentially Mackey complete topological *-algebra. If $a \in A$ and $S(e_A - a)$ is bounded in A, then there exists an element $b \in A$ such that $b^2 = a$ and $\beta_A(e_A - b) \leqslant 1$. In particular, when A satisfies condition* (c) *of Theorem* 2.7 *or condition*

(e) *a has only one square root in A,*

then b is self-adjoint if a is self-adjoint.

Corollary 2.9. *Let A be a unital sequentially Mackey complete topological *-algebra. If $a \in A$ is self-adjoint and $S(e_A - a)$ is bounded in A, then there exists a self-adjoint element $b \in A$ such that $b^2 = a$ and $\beta_A(e_A - b) \leqslant 1$.*

3. SOME RESULTS FOR TOPOLOGICAL *-ALGEBRAS

In the sequel we need the following result.

Proposition 3.1. *Let A be a commutative simplicial Gelfand–Mazur algebra with nonempty set $m(A)$. Then*

(i) $\mathrm{sp}^t_A(a) \setminus \{0\} \subset \{\varphi(a) : \varphi \in \mathrm{hom}A\} \subset \mathrm{sp}^t_A(a)$ *for each $a \in A$*

and

(ii) $\mathrm{sp}^t_A(a) = \{\varphi(a) : \varphi \in \mathrm{hom}A\}$ *for each $a \in A$ if A is a unital algebra.*

Proof. (i) Take $a \in A$ and $\mu \in \mathrm{sp}^t_A(a) \setminus \{0\}$. Then $\frac{a}{\mu} \notin \mathrm{Tqinv}A$. Therefore, the set

$$I = \mathrm{cl}\Big\{\frac{a}{\mu}b - b : b \in A\Big\}$$

cannot contain $\frac{a}{\mu}$, otherwise there is a net $(a_\lambda)_{\lambda\in\Lambda}$ in A such that $(\frac{a}{\mu}a_\lambda - a_\lambda)_{\lambda\in\Lambda}$ converges to $\frac{a}{\mu}$ in A or $(\frac{a}{\mu} \circ a_\lambda)_{\lambda\in\Lambda}$ converges to θ_A. This means that $\frac{a}{\mu} \in \mathrm{Tqinv}A$. Hence, $I \neq A$. Therefore, I is a closed regular ideal in A. Since A is simplicial, there exists a closed maximal regular ideal M in A such that $I \subset M$ and, since A is a Gelfand–Mazur algebra, then $M = \ker\varphi$ for a $\varphi \in \mathrm{hom}A$. Consequently, $\varphi(\frac{a}{\mu}b - b) = 0$ for each $b \in A$. Hence, $\varphi(a) = \mu$ (because φ is not trivial). This shows that

$$\mathrm{sp}^t_A(a) \setminus \{0\} \subset \{\varphi(a) : \varphi \in \mathrm{hom}A\}.$$

Let now $a \in A$ and $\mu = \varphi(a)$ for some $\varphi \in \mathrm{hom}A$. We must show that $\mu \in \mathrm{sp}^t_A(a)$. We suppose that $\mu \neq 0$ and $\mu \notin \mathrm{sp}^t_A(a)$. Then $\frac{a}{\mu} \in \mathrm{Tqinv}A$ and so there is a net $(c_\alpha)_{\alpha\in\mathscr{A}}$ such that $(\frac{a}{\mu} \circ c_\alpha)_{\alpha\in\mathscr{A}}$ converges to θ_A in A. Since φ is continuous, $(\frac{\varphi(a)}{\mu} \circ \varphi(c_\alpha))_{\alpha\in\mathscr{A}}$ converges to 0, but it is not possible, since $\varphi(a) = \mu$. Consequently, $\mu \in \mathrm{sp}^t_A(a)$ if $\mu \neq 0$.

Let now $\varphi(a) = 0$. If A does not have a unit, then automatically $0 \in \mathrm{sp}^t_A(a)$ and so $\varphi(a) \in \mathrm{sp}^t_A(a)$. If A has a unit and $0 \notin \mathrm{sp}^t_A(a)$, then $a \in \mathrm{Tqinv}A$. Therefore, there exists a net $(a_\beta)_{\beta\in\mathscr{B}}$ such that $(aa_\beta)_{\beta\in\mathscr{B}}$ converges to e_A in A. Then $(\varphi(a)\varphi(a_\beta))_{\beta\in\mathscr{B}}$ converges to 1. But this is impossible, since $\varphi(a) = 0$.

(ii) Let now A be a unital algebra. By statement (i), it is sufficient to show that $0 \in \mathrm{sp}^t_A(a)$ if and only if $\varphi(a) = 0$ for some $\varphi \in \mathrm{hom}A$.

Suppose first that $0 \in \mathrm{sp}^t_A(a)$. Then $a \notin \mathrm{Tinv}A$ and, similarly as above,

$$I = \mathrm{cl}\{ab : b \in A\}$$

is a closed ideal in A. Since A is a commutative unital simplicial Gelfand–Mazur algebra, there exists a $M \in m(A)$ such that $I \subset M = \ker \varphi$ for some $\varphi \in \hom A$. Therefore, $\varphi(a)\varphi(b) = 0$ for each $b \in A$. Again, since φ is not trivial, $\varphi(a) = 0$.

Suppose next that $\varphi(a) = 0$ for some $\varphi \in \hom A$. Then $a \notin \operatorname{Tinv} A$. Otherwise, there exists a net $(a_\lambda)_{\lambda\in\Lambda}$ such that $(aa_\lambda)_{\lambda\in\Lambda}$ converges to e_A in A. Then $(\varphi(a)\varphi(a_\lambda))_{\lambda\in\Lambda}$ converges to 1 contrary to $\varphi(a) = 0$. Consequently, in this case $0 \in \operatorname{sp}^t_A(a)$. □

Corollary 3.2. *Let A be a commutative advertive simplicial Gelfand–Mazur algebra with nonempty set* $m(A)$*. Then*

(i) $\operatorname{sp}_A(a) \setminus \{0\} \subset \{\varphi(a) : \varphi \in \hom A\} \subset \operatorname{sp}_A(a)$ *for each* $a \in A$

and

(ii) $\operatorname{sp}_A(a) = \{\varphi(a) : \varphi \in \hom A\}$ *for each* $a \in A$ *if A is an invertive algebra.*

Proof. In the present case $\operatorname{sp}_A(a) = \operatorname{sp}^t_A(a)$ for each $a \in A$. Therefore, the statements hold by Corollary 3.2. □

Corollary 3.2 (ii) was proved in [3], Proposition 5. Moreover, it was shown in [3], Proposition 6, that every topological algebra for which

$$\operatorname{sp}_A(a) = \{\varphi(a) : \varphi \in \hom A\}$$

for each $a \in A$ is an advertive algebra.

Proposition 3.3. *Let A be a unital sequentially Mackey complete locally pseudoconvex Hausdorff algebra for which* $\beta_A(a) = \rho_A(a)$ *for each* $a \in A$*. If, in addition, A satisfies the condition*[9]

(f) $\operatorname{sp}_A(a)$ *is a closed subset in* $\mathbb{C}$ *for each* $a \in A$ *with* $\operatorname{sp}_A(a) \subset (0,1)$*,*

then for every element $a \in A$ *with* $\operatorname{sp}_A(a) \subset (0,\infty)$*, there exists an element*[10] $b \in A$ *such that* $b^2 = a$*. In particular, when every maximal commutative unital subalgebra B of A is an invertive simplicial Gelfand–Mazur algebra, then* $\operatorname{sp}_A(b) \subset (0,\infty)$*.*

Proof. Let $a \in A$ be such that $\operatorname{sp}_A(a) \subset (0,\infty)$. If $\rho_A(e_A - a) < 1$, then $\beta_A(e_A - a) < 1$ by assumption. Therefore (see the proof of Theorem 2.1 and Corollary 2.5), there exists an element

$$b = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (a - e_A)^k \in A$$

such that $b^2 = a$ and $\rho_A(e_A - b) < 1$.

Let B be a maximal commutative unital subalgebra of A, containing a. If now B is an invertive simplicial Gelfand–Mazur algebra, then $\operatorname{Tinv} A = \operatorname{Inv} A$, $\hom B$ is not empty and

$$\{\varphi(a) : \varphi \in \hom B\} = \operatorname{sp}_B(a) = \operatorname{sp}_A(a) \subset (0,\infty)$$

for each $a \in B$ by Corollary 3.2. Therefore, $\varphi(a) > 0$ for each $\varphi \in \hom B$. Hence (by the formula (3), p. 361 from [33])

$$\varphi(b) = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} (\varphi(a) - 1)^k = \sqrt{\varphi(a)} > 0$$

for each $\varphi \in \hom B$. Consequently,

$$\operatorname{sp}_A(b) = \operatorname{sp}_B(b) = \{\varphi(b) : \varphi \in \hom B\} \subset (0,\infty).$$

[9] If A is a Q-algebra, then condition (f) is superfluous, because in this case the spectrum of every element of A is closed (see, for example, [29], Proposition 4.2).

[10] When $\beta_A(e_A - a) < 1$, then a has only one square root by Theorem 2.1.

Let again $a \in A$ be such that $\mathrm{sp}_A(a) \subset (0,\infty)$. Since $-1 \notin \mathrm{sp}_A(a)$, then $e_A + a \in \mathrm{Inv}A$. Let $c = (e_A + a)^{-1}$, $v = ac$ and B be a maximal commutative unital subalgebra of A, containing e_A, a, and c. Then $v = e_A - c \in B$ and $\mathrm{sp}_B(x) = \mathrm{sp}_A(x)$ for each $x \in B$. Since

$$\mathrm{sp}_A(c) = \left\{ \frac{1}{\mu + 1} : \mu \in \mathrm{sp}_A(a) \right\} \subset (0,1)$$

(see, for example, the equality (4.14) in [22]), then

$$\mathrm{sp}_A(v) = \mathrm{sp}_A(e_A - c) = 1 - \mathrm{sp}_A(c) \subset (0,1)$$

and

$$\mathrm{sp}_A(e_A - v) = 1 - \mathrm{sp}_A(v) \subset (0,1).$$

Therefore $\rho_A(e_A - c) < 1$ and $\rho_A(e_A - v) < 1$ by condition (f). Thus, by the first part of the proof, there exist $z, w \in A$ such that $z^2 = c$ and $w^2 = v$. Moreover, if B is a commutative invertive simplicial Gelfand–Mazur algebra, then

$$\mathrm{sp}_A(z) = \mathrm{sp}_B(z) = \{\varphi(z) : \varphi \in \hom B\} \subset (0,\infty)$$

and

$$\mathrm{sp}_A(w) = \mathrm{sp}_B(w) = \{\varphi(w) : \varphi \in \hom B\} \subset (0,\infty).$$

Taking this into account, $z, w \in \mathrm{Inv}A$, from $z^2 = c$ it follows that $(z^{-1})^2 = e_A + a$ and $b = z^{-1}w$ is the square root of a. Since

$$\mathrm{sp}_B(b) = \{\varphi(b) : \varphi \in \hom B\} = \{\varphi(z^{-1})\varphi(w) : \varphi \in \hom B\} = \left\{ \frac{\varphi(w)}{\varphi(z)} : \varphi \in \hom B \right\},$$

$\mathrm{sp}_B(z) \subset (0,\infty)$ and $\mathrm{sp}_B(w) \subset (0,\infty)$, then

$$\frac{\varphi(w)}{\varphi(z)} > 0$$

for each $\varphi \in \hom B$. Consequently, $\mathrm{sp}_A(b) = \mathrm{sp}_B(b) \subset (0,\infty)$. □

Corollary 3.4. *Let A be a unital complete locally m-(k-convex) Hausdorff algebra with bounded elements and $k \in (0,1]$. Then for every element $a \in A$ with $\mathrm{sp}_A(a) \subset (0,\infty)$ there exists an element*[11] *$b \in A$ such that $b^2 = a$ and $\mathrm{sp}_A(b) \subset (0,\infty)$.*

Proof. Take $a \in A$ and B a maximal commutative unital closed subalgebra of A, containing a. Then B is a commutative unital complete Hausdorff locally *m*-(*k*-convex) algebra with bounded elements. Therefore, $\beta_A(a) = \rho_A(b)$ for each $b \in B$ (see [6], Corollary 4.4) and $\mathrm{sp}_B(b)$ is a closed subset in $\mathbb{C}$ (see the proof of Proposition 3.2 in [7], pp. 203–204). Since $\mathrm{sp}_A(b) = \mathrm{sp}_B(b)$ and $\beta_A(b) = \beta_B(b)$ for each $b \in B$, then $\beta_A(a) = \rho_A(a)$ and condition (f) holds. Moreover, every maximal commutative unital (not necessarily closed) subalgebra of A is an invertive (by Corollary 2 in [3]) simplicial (by Corollary 5 in [5]) Gelfand–Mazur algebra (see, for example, [2], Corollary 2, or [8], Theorem 3.3). Hence, the result follows from Proposition 3.3. □

Theorem 3.5. *Let A be a unital sequentially Mackey complete topological *-algebra with continuous involution, for which $\beta_A(a) = \rho_A(a)$ for each $a \in A$. If, moreover, A satisfies the condition*[12]

(g) $\mathrm{sp}_A(a)$ *is a closed subset in $\mathbb{C}$ for each self-adjoint $a \in A$ with* $\mathrm{sp}_A(a) \subset (0,1)$,

then for every self-adjoint element $h \in A$ with $\mathrm{sp}_A(h) \subset (0,\infty)$ there exists a self-adjoint element[13] *$u \in A$ such that $u^2 = h$. In particular, when every maximal commutative unital *-subalgebra B of A is an invertive simplicial Gelfand–Mazur *-algebra, then $\mathrm{sp}_A(u) \subset (0,\infty)$.*

[11] See footnote 8.
[12] See footnote 7.
[13] See footnote 8.

Proof. The proof is similar to that of Proposition 3.3. Herewith, u is self-adjoint by Corollary 2.8. □

Corollary 3.6. *Let A be a unital complete locally m-(k-convex) Hausdorff *-algebra with continuous involution. If all elements of A are bounded, then for each self-adjoint element $h \in A$ with $\mathrm{sp}_A(h) \subset (0,\infty)$ there exists a self-adjoint element*[14] *$u \in A$ such that $u^2 = h$ and $\mathrm{sp}_A(u) \subset (0,\infty)$.*

Proof. The proof is similar to that of Corollary 3.4. □

Theorem 3.7. *Let A be a topological *-algebra in which every maximal commutative *-subalgebra is an advertive simplicial Gelfand–Mazur *-algebra and let $h_1,\ldots,h_n$ be self-adjoint elements in A such that $\mathrm{sp}_A(h_k) \subset [0,\infty)$ for each k with $1 \leqslant k \leqslant n$. Then*

$$\mathrm{sp}_A(h_1+\cdots+h_n) \subset [0,\infty).$$

Proof. Since $h_1+\cdots+h_n$ is self-adjoint,

$$\mathrm{sp}_A(h_1+\cdots+h_n) \subset [0,\infty).$$

Let B be a maximal commutative *-subalgebra of A which contains all elements $h_1,\ldots,h_n$. Since B is an advertive simplicial Gelfand–Mazur algebra, by assumption, $\hom B$ is not empty. Let λ be an arbitrary negative real number. It is known that

$$\mathrm{sp}_A(h_k)\cup\{0\} = \mathrm{sp}_B(h_k)\cup\{0\}$$

by Lemma 4.11 from [41], p. 47. As $\mathrm{sp}_B(h_k) \subset [0,\infty)$ for each k, it follows that $\frac{h_k}{\lambda} \in \mathrm{Qinv}B$ for each k. Therefore $\varphi(\frac{h_k}{\lambda}) \neq 1$ or $\varphi(h_k) \neq \lambda$ for each k. It means that $\varphi(h_k) \geqslant 0$ for each k. Hence

$$\varphi\Big(\frac{h_1+\cdots+h_n}{\lambda}\Big) = \frac{\varphi(h_1)+\cdots+\varphi(h_n)}{\lambda} \leqslant 0$$

for each $\varphi \in \hom B$. Namely by [3], p. 20,

$$\frac{h_1+\cdots+h_n}{\lambda} \in \mathrm{Tqinv}B = \mathrm{Qinv}B$$

for each $\lambda < 0$, since B is advertive. Consequently,

$$\mathrm{sp}_A(h_1+\cdots+h_n) \subset \mathrm{sp}_B(h_1+\cdots+h_n)\cup\{0\} \subset [0,\infty).$$ □

Corollary 3.8. *Let A be a complete locally m-pseudoconvex Hausdorff *-algebra and $h_1,\ldots,h_n$ be self-adjoint elements in A such that $\mathrm{sp}_A(h_k) \subset [0,\infty)$ for each k with $1 \leqslant k \leqslant n$. Then*

$$\mathrm{sp}_A(h_1+\cdots+h_n) \subset [0,\infty).$$

Proof. Let B be a maximal commutative closed *-subalgebra of A. Then B is a commutative complete locally m-pseudoconvex Hausdorff *-algebra. Since, as above, B is an advertive simplicial Gelfand–Mazur *-algebra, by Theorem 3.7 the proof is complete. □

---

[14] See footnote 8.

Remark 3.9. Notice that Theorem 2.1 for fundamental Fréchet algebras was proved partly in [10], Theorem 3.2, and for pseudocomplete locally convex algebras partly in [35], Lemma 1; Corollary 2.2 for complete locally m-pseudoconvex algebras (using projective limits of p-Banach algebras) was proved partly in [11], Theorem 5.3.4; Corollary 2.3 for complete locally m-convex algebras (using projective limits of Banach algebras) was proved in [38], Theorem 3.9, and in [22], Theorem 5.5.4; Corollary 2.4 for fundamental Fréchet algebras was proved partly in [10], Theorem 3.3; Corollary 2.5 for complete unital locally m-pseudoconvex algebras was proved partly in [11], Corollary 5.3.5, and for unital complete locally m-convex algebras in [22], Corollary 5.5.5, and partly in [25], Corollary 1.13; Corollary 3.2 for Banach algebras was proved in [41], Theorem 3.12, and for commutative locally m-convex Q-algebras in [29], pp. 74–76; Corollary 3.4 for complete unital locally m-convex algebras was proved in [39], Theorem 2.2, and in [22], Theorem 5.5.8; and Corollary 3.8 has been proved mostly for C*-algebras (see, for example, [37], Lemma 4.7.4, or [41], Lemma 6.4).

ACKNOWLEDGEMENTS

We gratefully acknowledge the financial support of the Estonian Science Foundation (grant 7320) and of the Estonian Targeted Financing Project (SF0180039s08).

REFERENCES

1. Abel, Mart. *Structure of Gelfand-Mazur Algebras*. Dissertation, University of Tartu, Tartu, 2003. Dissertationes Mathematicae Universitatis Tartuensis **31**. Tartu University Press, Tartu, 2003.
2. Abel, Mati. Gel'fand-Mazur algebras. In *Topological Vector Spaces, Algebras and Related Areas* (Hamilton, ON, 1994). Pitman Res. Notes Math. Ser. **316**. Longman Sci. Tech., Harlow, 1994, 116–129.
3. Abel, Mati. Advertive topological algebras. In *General Topological Algebras* (Tartu, 1999). Math. Stud. (Tartu), **1**. Est. Math. Soc., Tartu, 2001, 14–24.
4. Abel, Mati. Descriptions of the topological radical in topological algebras. In *General Topological Algebras* (Tartu, 1999). Math. Stud. (Tartu), **1**. Est. Math. Soc., Tartu, 2001, 25–31.
5. Abel, Mati. Inductive limits of Gelfand-Mazur algebras. *Int. J. Pure Appl. Math.*, 2004, **16**(3), 363–378.
6. Abel, Mati. Topological algebras with pseudoconvexly bounded elements. In *Topological Algebras, Their Applications, and Related Topics*. Banach Center Publ. **67**, Polish Acad. Sci., Warsaw, 2005, 21–33.
7. Abel, Mati. Structure of locally idempotent algebras. *Banach J. Math. Anal.*, 2007, **1**(2), 195–207.
8. Abel, Mati and Kokk, A. Locally pseudoconvex Gel'fand-Mazur algebras. *Eesti NSV Tead. Akad. Toim. Füüs.-Mat.*, 1988, **37**, 377–386 (in Russian).
9. Abel, Mati and Żelazko, W. Topologically invertible elements and topological spectrum. *Bull. Pol. Acad. Sci. Math.*, 2006, **54**(3–4), 257–271.
10. Ansari-Piri, E. and Zohri, A. The *n*th roots and the quasi square roots in complete metrizable fundamental topological algebras. *Far East J. Math. Sci.* (FJMS), 2008, **28**(3), 695–699.
11. Balachandran, V. K. *Topological Algebras*. North-Holland Mathematics Studies, 185. North-Holland Publishing Co., Amsterdam, 2000.
12. Beckensein, E., Narici, L., and Suffel, Ch. *Topological Algebras*. North-Holland Mathematics Studies, Vol. 24. Notas de Matemática, No. 60. [Mathematical Notes, No. 60] North-Holland Publishing Co., Amsterdam–New York–Oxford, 1977.
13. Bonsall, F. F. and Duncan, J. *Complete Normed Algebras*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 80. Springer-Verlag, New York–Heidelberg, 1973.
14. Chabboun, M. and El Kinani, A. Calcul fonctionnel holomorphe dans les algébres localement p-convexes. *Rend. Circ. Mat. Palermo* (2), 2005, **54**(2), 217–233.
15. Chabboun, M. and El Kinani, A. Calcul fonctionnel holomorphe dans les algébres localement p-convexes. *Bull. Greek Math. Soc.*, 2006, **52**, 77–90.
16. El Kinani, A. A note on harmonic calculus in m-convex algebras. *Rend. Istit. Mat. Univ. Trieste*, 2001, **33**(1–2), 105–112 (2002).
17. El Kinani, A. On harmonic functions operating in locally m-pseudoconvex algebras. *Bull. Belg. Math. Soc. Simon Stevin*, 2002, **9**(3), 447–454.
18. El Kinani, A. and Ifzarne, A. Calcul harmonique dans les algébres p-Banach involutives et applications. *Bull. Belg. Math. Soc. Simon Stevin*, 2001, **8**(4), 685–697.

19. El Kinani, A., Oubbi, L., and Oudadess, M. Spectral and boundedness radii in locally convex algebras. *Georgian Math. J.*, 1998, **5**(3), 233–241.
20. Ford, J. W. M. *Subalgebras of Banach Algebras Generated by Semigroups*. Thesis, Univ. of Newcastle on Tune, 1966.
21. Ford, J. W. M. A square root lemma for Banach *-algebras. *J. London Math. Soc.*, 1967, **42**, 521–522.
22. Fragoulopoulou, M. *Topological Algebras with Involution*. North-Holland Mathematics Studies, 200. Elsevier Science B.V., Amsterdam, 2005.
23. Hille, E. and Phillips, R. S. *Functional Analysis and Semi-Groups*. Rev. edn. American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I., 1957.
24. Hogbe-Nlend, H. *Bornologies and Functional Analysis. Introductory Course on the Theory of Duality Topology-Bornology and Its Use in Functional Analysis*. North-Holland Mathematics Studies, Vol. 26. Notas de Matemática, No. 62 [Notes on Mathematics, No. 62]. North-Holland Publishing Co., Amsterdam–New York–Oxford, 1977.
25. Husain, T. *Multiplicative Functionals on Topological Algebras*. Research Notes in Mathematics, 85. Pitman (Advanced Publishing Program), Boston, MA, 1983.
26. Husain, T. and Ringelhof, R. Representations of MQ^*-algebras. *Math. Ann.*, 1969, **180**, 297–306.
27. Jarchow, H. *Locally Convex Spaces*. Mathematische Leitfäden [Mathematical Textbooks]. B. G. Teubner, Stuttgart, 1981.
28. Luha, K. Topological invertibility in topological algebras. In *Application of Topology in Algebra and Differential Geometry (Tartu, 1991)*. *Tartu Ülik. Toim.*, 1992, **940**, 71–74.
29. Mallios, M. *Topological Algebras. Selected Topics*. North-Holland Mathematics Studies, 124. Notas de Matemática, 109. North-Holland Publishing Co., Amsterdam, 1986.
30. Michael, E. A. Locally multiplicatively-convex topological algebras. *Mem. Amer. Math. Soc.*, 1952, No. 11.
31. Oubbi, L. Further radii in topological algebras. *Bull. Belg. Math. Soc. Simon Stevin*, 2002, **9**(2), 279–292.
32. Oubbi, L. A Kaplansky-Meyer theorem for subalgebras. *Bull. Belg. Math. Soc. Simon Stevin*, 2009, **16**(2), 305–312.
33. Palmer, Th. W. *Banach Algebras and the General Theory of *-Algebras*. Vol. I. *Algebras and Banach Algebras*. Encyclopedia of Mathematics and its Applications, 49. Cambridge University Press, Cambridge, 1994.
34. Palmer, Th. W. *Banach Algebras and the General Theory of $*$-Algebras*. Vol. 2. *$*$-Algebras*. Encyclopedia of Mathematics and its Applications, 79. Cambridge University Press, Cambridge, 2001.
35. Powell, J. D. Representations of locally convex *-algebras. *Proc. Amer. Math. Soc.*, 1974, **44**, 341–346.
36. Prudnikov, A. P., Brychkov, Yu. A., and Marichev, O. I. *Integrals and Series. Elementary Functions*. Nauka, Moscow, 1981 (in Russian).
37. Rickart, C. E. *General Theory of Banach Algebras*. D. van Nostrand Co., Inc., Princeton, N.J.–Toronto–London–New York, 1960.
38. Štěrbová, D. Square roots and quasi-square roots in locally multiplicatively convex algebras. *Sb. Prací Přírodověd. Fak. Univ. Palackého v Olomouci Mat.*, 1980, **19**, 103–110.
39. Štěrbová, D. Square roots of elements with an unbounded spectrum. *Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math.*, 1984, **23**, 103–106.
40. Williamson, J. H. On topologising the field $C(t)$. *Proc. Amer. Math. Soc.*, 1954, **5**, 729–734.
41. Yood, B. *Banach Algebras – An Introduction*. Carleton-Ottawa Math. Lecture Notes Series 9. Carleton Univ. Press, Carleton–Ottawa, 1988.
42. Żelazko, W. *Selected Topics in Topological Algebras. Lectures 1969/1970.* Lecture Notes Series, No. 31. Matematisk Institut, Aarhus Universitet, Aarhus, 1971.

Fordi lemma topoloogiliste *-algebrate korral

Mart Abel ja Mati Abel

On tõestatud Fordi lemma analoog teatud liiki topoloogiliste algebrate (erijuhul topoloogiliste *-algebrate) jaoks ja saadud tulemusi on kasutatud topoloogiliste *-algebrate omaduste kirjeldamisel.