Proceedings of the Estonian Academy of Sciences,
2011, **60**, 2, 115–120
doi: 10.3176/proc.2011.2.06
Available online at www.eap.ee/proceedings

MATHEMATICS

A study on generalized absolute summability factors for a triangular matrix

Ekrem Savaş

Department of Mathematics, Istanbul Ticaret University, Üsküdar-Istanbul, Turkey; ekremsavas@yahoo.com; esavas@iticu.edu.tr

Received 3 August 2009, revised 26 March 2010, accepted 5 April 2010

Abstract. In this paper, we establish a summability factor theorem for summability $|A,\delta|_k$. This paper is an extension of the main result of Savas, E. A study on absolute summability factors for a triangular matrix. *Math. Ineq. Appl.*, 2009, **12**(19), 141–146.

Key words: absolute summability, summability factors, triangular matrix.

1. INTRODUCTION

Savas [2] obtained sufficient conditions for $\sum a_n\lambda_n$ to be $|A|_k$-summable, $k\in\mathbb{N}$. In this paper, a theorem on $|A,\delta|_k$-summability methods is proved. This theorem includes a known result.

Let A be a lower triangular matrix, $\{s_n\}$ a sequence. Then

$$A_n := \sum_{\nu=0}^{n} a_{n\nu}s_\nu .$$

A series $\sum a_n$ is said to be summable $|A|_k, k\geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1}|A_n - A_{n-1}|^k < \infty, \tag{1}$$

and it is said to be summable $|A,\delta|_k, k\geq 1$ and $\delta\geq 0$ if (see, [1])

$$\sum_{n=1}^{\infty} n^{\delta k+k-1}|A_n - A_{n-1}|^k < \infty. \tag{2}$$

We may associate with A two lower triangular matrices $\overline{A}$ and $\hat{A}$ defined as follows:

$$\bar{a}_{n\nu} = \sum_{r=\nu}^{n} a_{nr}, \qquad n,\nu = 0,1,2,\ldots,$$

and

$$\hat{a}_{n\nu} = \bar{a}_{n\nu} - \bar{a}_{n-1,\nu}, \qquad n = 1,2,3,\ldots .$$

Also we shall define

$$y_n := \sum_{i=0}^{n} a_{ni} s_i = \sum_{i=0}^{n} a_{ni} \sum_{\nu=0}^{i} \lambda_\nu a_\nu$$
$$= \sum_{\nu=0}^{n} \lambda_\nu a_\nu \sum_{i=\nu}^{n} a_{ni} = \sum_{\nu=0}^{n} \bar{a}_{n\nu} \lambda_\nu a_\nu$$

and

$$Y_n := y_n - y_{n-1} = \sum_{\nu=0}^{n} (\bar{a}_{n\nu} - \bar{a}_{n-1,\nu}) \lambda_\nu a_\nu = \sum_{\nu=0}^{n} \hat{a}_{n\nu} \lambda_\nu a_\nu. \quad (3)$$

Given any sequence $\{x_n\}$, the notation $x_n \asymp O(1)$ means $x_n = O(1)$ and $1/x_n = O(1)$. For any matrix entry $a_{n\nu}$, $\Delta_\nu a_{n\nu} := a_{n\nu} - a_{n\nu+1}$.

2. MAIN RESULT

Theorem 1. *Let A be a lower triangular matrix with nonnegative entries such that*

(i) $\bar{a}_{n0} = 1, n = 0, 1, \ldots,$

(ii) $a_{n-1,\nu} \geq a_{n\nu}$ for $n \geq \nu + 1,$

(iii) $na_{nn} \asymp O(1)$, *and*

(iv) $\sum_{\nu=1}^{n-1} a_{\nu\nu} |\hat{a}_{n,\nu+1}| = O(a_{nn})$.

(v) $\displaystyle\sum_{n=\nu+1}^{m+1} n^{\delta k} |\Delta_\nu \hat{a}_{n\nu}| = O\left(\nu^{\delta k} a_{\nu\nu}\right)$ *and*

(vi) $\displaystyle\sum_{n=\nu+1}^{m+1} n^{\delta k} \hat{a}_{n\nu+1} = O\left(\nu^{\delta k}\right).$

Let t_n^1 denote the nth $(C,1)$ mean of $\{na_n\}$. If

(vii) $\displaystyle\sum_{\nu=1}^{\infty} \nu^{\delta k} a_{\nu\nu} |\lambda_\nu|^k |t_\nu^1|^k = O(1),$

(viii) $\displaystyle\sum_{\nu=1}^{\infty} \nu^{\delta k} |a_{\nu\nu}|^{1-k} |\Delta\lambda_\nu|^k |t_\nu^1|^k = O(1),$

(ix) $\displaystyle\sum_{\nu=1}^{\infty} \nu^{\delta k} a_{\nu\nu} |\lambda_{\nu+1}|^k |t_\nu^1|^k = O(1),$

then the series $\sum a_n \lambda_n$ is summable $|A,\delta|_k, k \geq 1, 0 \leq \delta < 1/k$.

Proof. From (i) it follows that $\hat{a}_{n,0} = 0$. Also $(\hat{a}_{n1}\lambda_1)a_1$ is bounded.

Using (3) we may write, for example:

$$
\begin{aligned}
Y_n &= \sum_{\nu=2}^{n} \left(\frac{\hat{a}_{n\nu}\lambda_\nu}{\nu}\right) \nu a_\nu \\
&= \sum_{\nu=2}^{n} \left(\frac{\hat{a}_{n\nu}\lambda_\nu}{\nu}\right) \left[\sum_{r=1}^{\nu} r a_r - \sum_{r=1}^{\nu-1} r a_r\right] \\
&= \sum_{\nu=2}^{n-1} \Delta_\nu \left(\frac{\hat{a}_{n\nu}\lambda_\nu}{\nu}\right) \sum_{r=1}^{\nu} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{\nu=2}^{n} \nu a_\nu \\
&= \sum_{\nu=2}^{n-1} (\Delta_\nu \hat{a}_{n\nu}) \lambda_\nu \frac{\nu+1}{\nu} t_\nu^1 + \sum_{\nu=2}^{n-1} \hat{a}_{n,\nu+1} (\Delta\lambda_\nu) \frac{\nu+1}{\nu} t_\nu^1 \\
&\quad + \sum_{\nu=2}^{n-1} \hat{a}_{n,\nu+1} \lambda_{\nu+1} \frac{1}{\nu} t_\nu^1 + \frac{(n+1)a_{nn}\lambda_n t_n^1}{n} \\
&= T_{n1} + T_{n2} + T_{n3} + T_{n4}.
\end{aligned}
$$

In order to prove our theorem it is sufficient, by Minkowski's inequality, to show that

$$
\sum_{n=2}^{\infty} n^{\delta k+k-1} |T_{nr}|^k < \infty, \quad \text{for} \quad r = 1,2,3,4.
$$

Using Hölder's inequality, (iii), (v), and (vii)

$$
\begin{aligned}
I_1 := \sum_{n=2}^{m+1} n^{\delta k+k-1} |T_{n1}|^k &= \sum_{n=2}^{m+1} n^{\delta k+k-1} \left| \sum_{\nu=1}^{n-1} \Delta_\nu \hat{a}_{n\nu} \lambda_\nu \frac{\nu+1}{\nu} t_\nu^1 \right|^k \\
&= O(1) \sum_{n=2}^{m+1} n^{\delta k+k-1} \left(\sum_{\nu=1}^{n-1} |\Delta_\nu \hat{a}_{n\nu}| |\lambda_\nu| |t_\nu^1| \right)^k \\
&= O(1) \sum_{n=2}^{m+1} n^{\delta k+k-1} \left(\sum_{\nu=1}^{n-1} |\Delta_\nu \hat{a}_{n\nu}| |\lambda_\nu|^k |t_\nu^1|^k \right) \left(\sum_{\nu=1}^{n-1} |\Delta_\nu \hat{a}_{n\nu}| \right)^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} n^{\delta k} (n a_{nn})^{k-1} \sum_{\nu=1}^{n-1} |\lambda_\nu|^k |t_\nu^1|^k |\Delta_\nu \hat{a}_{n\nu}| \\
&= O(1) \sum_{\nu=1}^{m} |\lambda_\nu|^k |t_\nu^1|^k \sum_{n=\nu+1}^{m+1} n^{\delta k} (n a_{nn})^{k-1} |\Delta_\nu \hat{a}_{n\nu}| \\
&= O(1) \sum_{\nu=1}^{m} |\lambda_\nu|^k |t_\nu^1|^k \sum_{n=\nu+1}^{m+1} n^{\delta k} |\Delta_\nu \hat{a}_{n\nu}| \\
&= O(1) \sum_{\nu=1}^{m} \nu^{\delta k} a_{\nu\nu} |\lambda_\nu|^k |t_\nu^1|^k \\
&= O(1).
\end{aligned}
$$

Using Hölder's inequality, (iii), (iv), (vi), and (viii)

$$\begin{aligned}
I_2 :&= \sum_{n=2}^{m+1} n^{\delta k+k-1}|T_{n2}|^k = \sum_{n=2}^{m+1} n^{\delta k+k-1}\left|\sum_{\nu=1}^{n-1}\hat{a}_{n,\nu+1}(\Delta\lambda_\nu)\frac{\nu+1}{\nu}t_\nu^1\right|^k \\
&\leq \sum_{n=2}^{m+1} n^{\delta k+k-1}\left[\sum_{\nu=1}^{n-1}\hat{a}_{n,\nu+1}|\Delta\lambda_\nu|\frac{\nu+1}{\nu}|t_\nu^1|\right]^k \\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\left[\sum_{\nu=1}^{n-1}|\Delta\lambda_\nu||t_\nu^1||\hat{a}_{n,\nu+1}|\right]^k \\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\left[\sum_{\nu=1}^{n-1}|\Delta\lambda_\nu|^k|t_\nu^1|^k|a_{\nu\nu}|^{1-k}|\hat{a}_{n,\nu+1}|\right]\left[\sum_{\nu=1}^{n-1}|a_{\nu\nu}||\hat{a}_{n,\nu+1}|\right]^{k-1} \\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{\nu=1}^{n-1}|a_{\nu\nu}|^{1-k}||\Delta\lambda_\nu|^k|t_\nu^1|^k|\hat{a}_{n,\nu+1}| \\
&= O(1)\sum_{\nu=1}^{m}|a_{\nu\nu}|^{1-k}|\Delta\lambda_\nu|^k|t_\nu^1|^k\sum_{n=\nu+1}^{m+1} n^{\delta k}(na_{nn})^{k-1}\hat{a}_{n,\nu+1} \\
&= O(1)\sum_{\nu=1}^{m}|a_{\nu\nu}|^{1-k}|\Delta\lambda_\nu|^k|t_\nu^1|^k\sum_{n=\nu+1}^{m+1} n^{\delta k}\hat{a}_{n,\nu+1} \\
&= O(1)\sum_{\nu=1}^{m}\nu^{\delta k}|a_{\nu\nu}|^{1-k}|\Delta\lambda_\nu|^k|t_\nu^1|^k \\
&= O(1).
\end{aligned}$$

Using Hölder's inequality, (iii), (iv), (vi), and (ix)

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k+k-1}|T_{n3}|^k &= \sum_{n=2}^{m+1} n^{\delta k+k-1}\left|\sum_{\nu=1}^{n-1}\frac{\hat{a}_{n,\nu+1}\lambda_{\nu+1}}{\nu}t_\nu^1\right|^k \\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\left[\sum_{\nu=1}^{n-1}|a_{\nu\nu}||\lambda_{\nu+1}||\hat{a}_{n,\nu+1}||t_\nu^1|\right]^k \\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k+k-1}\left[\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}|^k|a_{\nu\nu}||t_\nu^1|^k|\hat{a}_{n,\nu+1}|\right]\left[\sum_{\nu=1}^{n-1}|a_{\nu\nu}||\hat{a}_{n,\nu+1}|\right]^{k-1} \\
&= O(1)\sum_{n=2}^{m+1} n^{\delta k}(na_{nn})^{k-1}\sum_{\nu=1}^{n-1}|\lambda_{\nu+1}|^k|a_{\nu\nu}||t_\nu^1|^k|\hat{a}_{n,\nu+1}| \\
&= O(1)\sum_{\nu=1}^{m}|\lambda_{\nu+1}|^k|a_{\nu\nu}||t_\nu^1|^k\sum_{n=\nu+1}^{m+1} n^{\delta k}\hat{a}_{n,\nu+1} \\
&= O(1)\sum_{\nu=1}^{m}\nu^{\delta k}a_{\nu\nu}|\lambda_{\nu+1}|^k|t_\nu^1|^k \\
&= O(1).
\end{aligned}$$

Finally, using (iii) and (v)

$$\begin{aligned}\sum_{n=1}^{m} n^{\delta k+k-1}|T_{n4}|^k &= \sum_{n=1}^{m} n^{\delta k+k-1}\left|\frac{(n+1)a_{nn}\lambda_n t_n^1}{n}\right|^k \\ &= O(1)\sum_{n=1}^{m} n^{\delta k}(na_{nn})^{k-1}a_{nn}|\lambda_n|^k|t_n^1|^k \\ &= O(1)\sum_{n=1}^{m} n^{\delta k}a_{nn}|\lambda_n|^k|t_n^1|^k \\ &= O(1). \qquad \square\end{aligned}$$

Setting $\delta = 0$ in the theorem yields the following corollary:

Corollary 1 ([2]). *Let A be a triangle satisfying conditions (i)–(iv) of Theorem 1 and let t_n^1 denote the nth $(C,1)$ mean of $\{na_n\}$. If*

(v) $\sum_{v=1}^{\infty} a_{vv}|\lambda_v|^k|t_v^1|^k = O(1)$,

(vi) $\sum_{v=1}^{\infty} |a_{vv}|^{1-k}|\Delta\lambda_v|^k|t_v^1|^k = O(1)$,

(vii) $\sum_{v=1}^{\infty} a_{vv}|\lambda_{v+1}|^k|t_v^1|^k = O(1)$,

then the series $\sum a_n\lambda_n$ is summable $|A|_k$, $k \in \mathbb{N}$.

Remark. I must note that in the theorem of Savas [2], the following condition should be added.

(vii) $\sum_{v=1}^{\infty} a_{vv}|\lambda_{v+1}|^k|t_v^1|^k = O(1)$.

3. CONCLUSION

Let $\sum a_v$ denote a series with partial sums s_n. For an infinite matrix T, the nth term of the T-transform of $\{s_n\}$ is denoted by

$$t_n = \sum_{v=0}^{\infty} t_{nv}s_v.$$

Let σ_n^α denote the nth terms of the transform of a Cesáro matrix (C,α) of a sequence $\{s_n\}$. In 1957 Fleet [1] gave the following definition. A series $\sum a_n$, with partial sums s_n, is said to be absolutely (C,α) summable of order $k \geq 1$, written $\sum a_n$ is summable $|C,\alpha|_k$, if

$$\sum_{n=1}^{\infty} n^{k-1}\left|\sigma_{n-1}^\alpha - \sigma_n^\alpha\right|^k < \infty. \tag{4}$$

Recently, Savas [2] obtained an absolute summability factor theorem for lower triangular matrices. A summability factor theorem for summability $|A,\delta|_k$ has not been studied so far. The present paper fills up a gap in the existing literature.

ACKNOWLEDGEMENTS

I wish to thank the referees for their careful reading of the manuscript and for their helpful suggestions.

REFERENCES

1. Fleet, T. M. On an extension of absolute summability and some theorems of Littlewood and Paley. *Proc. London Math. Soc.*, 1957, **3**(7), 113–141.
2. Savas, E. A study on absolute summability factors for a triangular matrix. *Math. Ineq. Appl.*, 2009, **12**(19), 141–146.

Kolmnurksete maatriksmenetluste üldistatud absoluutse summeeruvuse teguritest

Ekrem Savaş

Olgu A kolmnurkne maatriks $k \geq 1$ ja $\delta \geq 0$. Artiklis on defineeritud maatriksmenetlusega A k-järku üldistatud absoluutse summeeruvuse ehk $|A,\delta|_k$-summeeruvuse mõiste ja leitud piisavad tingimused selleks, et rida $\Sigma\lambda_n a_n$ oleks $|A,\delta|_k$-summeeruv, st et arvud λ_n oleksid menetluse A k-järku üldistatud absoluutse summeeruvuse teguriteks. Saadud tulemus üldistab autori varasemat tulemust (vt Savas, E. *Math. Ineq. Appl.*, 2009, **12**(19), 141–146).