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MATHEMATICS

Convergence theorems on generalized equilibrium problems and fixed point problems with applications

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Abstract. In this paper, we introduce an iterative algorithm for finding a common element in the set of solutions to generalized equilibrium problems and a set of fixed points of strict pseudo-contractions. Strong convergence theorems are established in the framework of Hilbert spaces. The results presented in this paper mainly improve on the corresponding results reported by many others.

Key words: iterative algorithm, variational inequality, equilibrium problem, nonexpansive mapping, fixed point.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we always assume that C is a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a nonlinear mapping. Recall the following definitions.

(a) A is said to be *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

(b) A is said to be *strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, T is said to be α*-strongly-monotone*.

(c) A is said to be *inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, A is said to be α*-inverse-strongly monotone*.

Recall that the classical variational inequality problem, denoted by $VI(C,A)$, is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

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Given $z \in H$ and $u \in C$, we see that the following inequality holds

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in C,$$

if and only if $u = P_C z$, where P_C denotes the metric projection from H onto C. From the above we see that $u \in C$ is a solution to problem (1.1) if and only if u satisfies the following equation:

$$u = P_C(u - \rho Tu), \tag{1.2}$$

where $\rho > 0$ is a constant. This implies that problem (1.1) and problem (1.2) are equivalent. This alternative formula is very important form the numerical analysis point of view.

Let $T : C \to C$ be a nonlinear mapping. In this paper, we use $F(T)$ to denote the set of fixed points of T. Recall the following definitions.

(d) The mapping T is said to be *contractive* if there exists a constant $\alpha \in (0,1)$ such that

$$\|Tx - Ty\| \leq \alpha \|x - y\|, \quad \forall x, y \in C.$$

(e) The mapping T is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(f) T is said to be *strictly pseudo-contractive* with the coefficient $k \in [0,1)$ if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

For such a case, T is also said to be a *k-strict pseudo-contraction*.

(g) T is said to be *pseudo-contractive* if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \quad \forall x, y \in C.$$

Clearly, the class of strict pseudo-contractions falls into the one between the classes of non-expansive mappings and pseudo-contractions.

Let $A : C \to H$ be an α-inverse-strongly monotone mapping, F a bi-function of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. We consider the following generalized equilibrium problem.

$$\text{Find } x \in C \text{ such that } F(x,y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F,A)$, i.e.,

$$EP(F,A) = \{x \in C : F(x,y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C\}.$$

Next, we give some special cases of problem (1.3).

(i) If $A \equiv 0$, the zero mapping, then problem (1.3) is reduced to the the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x,y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F)$, i.e.,

$$EP(F) = \{x \in C : F(x,y) \geq 0, \quad \forall y \in C\}.$$

(ii) If $F \equiv 0$, then problem (1.3) is reduced to the classical variational inequality problem (1.1).

Problem (1.3) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, mini-max problems, the Nash equilibrium problem in noncooperative games, and others; see, for instance, [1,12].

To study the equilibrium problems (1.3) and (1.4), we may assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x,y);$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semi-continuous.

Recently, Takahashi and Takahashi [23] considered problem (1.4) by an iterative method. To be more precise, they proved the following theorem.

Theorem TT1. *Let C be a nonempty closed convex subset of H. Let F be a bi-function from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$ and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(f) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{cases} F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n \rangle \geq 0, \quad \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S y_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\} \in [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy

$$\lim_{n\to\infty} \alpha_n = 0, \; \sum_{n=1}^{\infty} \alpha_n = \infty, \; \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

$$\liminf_{n\to\infty} r_n > 0, \; and \; \sum_{n=1}^{\infty} ||r_{n+1} - r_n|| < \infty.$$

Then $\{x_n\}$ and $\{y_n\}$ converge strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S)\cap EP(F)} f(z)$.

Very recently, Takahashi and Takahashi [24] further considered the generalized equilibrium problem (1.3). They obtained the following result in a real Hilbert space.

Theorem TT2. *Let C be a closed convex subset of a real Hilbert space H and let $F : C \times C \to \mathbb{R}$ be a bi-function satisfying $(A1)$, $(A2)$, $(A3)$, and $(A4)$. Let A be an α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap EP(F,A) \neq \emptyset$. Let $u \in C$ and $x_1 \in C$ and let $\{z_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by*

$$\begin{cases} F(z_n, y) + \langle Ax_n, y - z_n \rangle + \frac{1}{r}\langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S[\alpha_n u + (1-\alpha_n) z_n], \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset [0,1]$, and $\{r_n\} \subset [0, 2\alpha]$, satisfy

$$0 < c \leq \beta_n \leq d < 1, \quad 0 < a\lambda_n \leq b < 2\alpha,$$

$$\lim_{n\to\infty} (\lambda_n - \lambda_{n+1}) = 0, \; \lim_{n\to\infty} \alpha_n = 0, \; and \; \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then, $\{x_n\}$ converges strongly to $z = P_{F(S)\cap EP(F,A)} u$.

In this paper, motivated by the research going on in this direction [4,5,7,9,10,13–17,19,20,22–25,27], we introduce a general iterative algorithm for the problem of finding a common element in the set of solutions to problem (1.3) and the set of fixed points of a strict pseudo-contraction. Strong convergence theorems are established in the framework of Hilbert spaces. The results presented in this paper improve and extend the corresponding results reported by many others.

In order to prove our main results, we need the following lemmas.

The following lemma can be found in [1] and [9].

Lemma 1.1. *Let C be a nonempty closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bi-function satisfying $(A1)-(A4)$. Then, for any $r > 0$ and $x \in H$ there exists $z \in C$ such that*

$$F(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \{z \in C : F(z,y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C\}$$

for all $r > 0$ and $x \in H$. Then, the following hold:
(1) *T_r is single-valued;*
(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x-y\rangle;$$

(3) *$F(T_r) = EP(F)$;*
(4) *$EP(F)$ is closed and convex.*

Lemma 1.2 ([21]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with*

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leq 0.$$

Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 1.3 ([3]). *Let C be a closed convex subset of a strictly convex Banach space E. Let $\{T_n : n \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on C. Suppose that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty}\lambda_n = 1$. Then a mapping S on C defined by*

$$Sx = \sum_{n=1}^{\infty}\lambda_n T_n x$$

for $x \in C$ is well defined, nonexpansive and $F(S) = \cap_{n=1}^{\infty} F(T_n)$ holds.

Lemma 1.4 ([2]). *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E, and $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero.*

Lemma 1.5 ([26]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$;
(ii) $\limsup_{n\to\infty}\delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.
Then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 1.6 ([28]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to H$ a k-strict pseudo-contraction with a fixed point. Then $F(T)$ is closed and convex. Define $S : C \to H$ by $Sx = kx + (1-k)Tx$ for each $x \in C$. Then S is nonexpansive such that $F(S) = F(T)$.*

2. MAIN RESULTS

Theorem 2.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let F_1 and F_2 be two bi-functions from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4), *respectively. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and $B : C \to H$ a β-inverse-strongly monotone mapping. Let $T : C \to C$ be a k-strict pseudo-contraction with a fixed point. Define a mapping $S : C \to C$ by $Sx = kx + (1-k)Tx$, $\forall x \in C$. Assume that $F = EP(F_1, A) \cap EP(F_2, B) \cap F(T) \neq \emptyset$. Let $u \in C$, $x_1 \in C$, and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} F_1(u_n, u) + \langle Ax_n, u - u_n \rangle + \frac{1}{r}\langle u - u_n, u_n - x_n \rangle \geq 0, \quad \forall u \in C, \\ F_2(v_n, v) + \langle Bx_n, v - v_n \rangle + \frac{1}{s}\langle v - v_n, v_n - x_n \rangle \geq 0, \quad \forall v \in C, \\ y_n = \gamma_n u_n + (1-\gamma_n) v_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S[\alpha_n u + (1-\alpha_n) y_n], \quad \forall n \geq 1, \end{cases} \tag{Υ}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$, $r \in (0, 2\alpha)$, and $s \in (0, 2\beta)$. If the above control sequences satisfy the following restrictions
(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(c) $\lim_{n\to\infty} \gamma_n = \gamma \in (0,1)$,
then the sequence $\{x_n\}$ defined by the iterative algorithm (Υ) will converge strongly to $z \in F$, where $z = P_F u$.

Proof. The proof is divided into five steps.

Step 1. Show that the sequence $\{x_n\}$ is bounded.

First, we claim that the mappings $I - rA$ and $I - sB$ are nonexpansive. Indeed, for each $x, y \in C$, we have

$$\begin{aligned} \|(I-rA)x - (I-rA)y\|^2 &= \|x - y - r(Ax - Ay)\|^2 \\ &= \|x-y\|^2 - 2r\langle x-y, Ax-Ay\rangle + r^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 - 2r\alpha\|Ax-Ay\|^2 + r^2\|Ax-Ay\|^2 \\ &= \|x-y\|^2 - r(2\alpha - r)\|Ax-Ay\|^2. \end{aligned}$$

It follows from the condition $r \in (0, 2\alpha)$ that the mapping $I - rA$ is nonexpansive, so is $I - sB$. Note that u_n can be rewritten as $u_n = T_r(I - rA)x_n$ and v_n can be rewritten as $v_n = T_s(I - sB)x_n$ for each $n \geq 1$. Let $p \in F$. It follows from Lemma 1.1 that

$$p = T_r(I - rA)p = T_s(I - sB)p = Tp.$$

Notice that

$$\begin{aligned} \|y_n - p\| &= \|\gamma_n u_n + (1-\gamma_n)v_n - p\| \\ &\leq \gamma_n\|u_n - p\| + (1-\gamma_n)\|v_n - p\| \\ &= \gamma_n\|T_r(I-rA)x_n - T_r(I-rA)p\| + (1-\gamma_n)\|T_s(I-sB)x_n - T_s(I-sB)p\| \\ &\leq \gamma_n\|(I-rA)x_n - (I-rA)p\| + (1-\gamma_n)\|(I-sB)x_n - (I-sB)p\| \\ &\leq \gamma_n\|x_n - p\| + (1-\gamma_n)\|x_n - p\| \\ &= \|x_n - p\|. \end{aligned}$$

From Lemma 1.6, we see that S is a nonexpansive mapping with $F(T)=F(S)$. It follows that

$$\begin{aligned}\|x_{n+1}-p\| &= \|\beta_n x_n+(1-\beta_n)S[\alpha_n u+(1-\alpha_n)y_n]-p\|\\ &\le \beta_n\|x_n-p\|+(1-\beta_n)\|S[\alpha_n u+(1-\alpha_n)y_n]-p\|\\ &\le \beta_n\|x_n-p\|+(1-\beta_n)\|[\alpha_n u+(1-\alpha_n)y_n]-p\|\\ &\le \beta_n\|x_n-p\|+(1-\beta_n)\alpha_n\|u-p\|+(1-\beta_n)(1-\alpha_n)\|y_n-p\|\\ &\le [1-\alpha_n(1-\beta_n)]\|x_n-p\|+(1-\beta_n)\alpha_n\|u-p\|.\end{aligned}$$

Putting $M_1=\max\{\|x_1-p\|,\|u-p\|\}$, we have that $\|x_n-p\|\le M_1$ for all $n\ge 1$. Indeed, we can easily see that $\|x_1-p\|\le M_1$. Suppose that $\|x_k-p\|\le M_1$ for some k. Then, we have that

$$\|x_{k+1}-p\|\le[1-\alpha_k(1-\beta_k)]M_1+(1-\beta_k)\alpha_k M_1=M_1.$$

This shows that $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{u_n\}$, and $\{v_n\}$.

Step 2. Show that $x_{n+1}-x_n\to 0$ as $n\to\infty$.

Note that

$$\begin{aligned}y_{n+1}-y_n &= \gamma_{n+1}u_{n+1}+(1-\gamma_{n+1})v_{n+1}-[\gamma_n u_n+(1-\gamma_n)v_n]\\ &= \gamma_{n+1}(u_{n+1}-u_n)+(\gamma_{n+1}-\gamma_n)(u_n-v_n)+(1-\gamma_{n+1})(v_{n+1}-v_n).\end{aligned}$$

It follows that

$$\begin{aligned}\|y_{n+1}-y_n\| &\le \gamma_{n+1}\|u_{n+1}-u_n\|+(1-\gamma_{n+1})\|v_{n+1}-v_n\|+|\gamma_{n+1}-\gamma_n|\|u_n-v_n\|\\ &= \gamma_{n+1}\|T_r x_{n+1}-T_r x_n\|+(1-\gamma_{n+1})\|T_s x_{n+1}-T_s x_n\|+|\gamma_{n+1}-\gamma_n|\|u_n-v_n\|\\ &\le \gamma_{n+1}\|x_{n+1}-x_n\|+(1-\gamma_{n+1})\|x_{n+1}-x_n\|+|\gamma_{n+1}-\gamma_n|\|u_n-v_n\|\\ &= \|x_{n+1}-x_n\|+|\gamma_{n+1}-\gamma_n|M_2, \qquad (2.1)\end{aligned}$$

where M_2 is an appropriate constant such that $M_2\ge\sup_{n\ge1}\{\|u_n-v_n\|\}$. Put

$$\rho_n=\alpha_n u+(1-\alpha_n)y_n,\quad \forall n\ge 1.$$

Notice that

$$\begin{aligned}\rho_{n+1}-\rho_n &= \alpha_{n+1}u+(1-\alpha_{n+1})y_{n+1}-[\alpha_n u+(1-\alpha_n)y_n]\\ &= (\alpha_{n+1}-\alpha_n)(u-y_n)+(1-\alpha_{n+1})(y_{n+1}-y_n).\end{aligned}$$

It follows from (2.1) that

$$\begin{aligned}\|\rho_{n+1}-\rho_n\| &\le |\alpha_{n+1}-\alpha_n|\|u-y_n\|+(1-\alpha_{n+1})\|y_{n+1}-y_n\|\\ &\le |\alpha_{n+1}-\alpha_n|\|u-y_n\|+\|x_{n+1}-x_n\|+|\gamma_{n+1}-\gamma_n|M_2.\end{aligned}$$

This implies that

$$\begin{aligned}\|\rho_{n+1}-\rho_n\|-\|x_{n+1}-x_n\| &\le |\alpha_{n+1}-\alpha_n|\|u-y_n\|+|\gamma_{n+1}-\gamma_n|M_2\\ &\le |\alpha_{n+1}-\alpha_n|M_3+|\gamma_{n+1}-\gamma_n|M_2,\end{aligned}$$

where M_3 is an appropriate constant such that $M_3\ge\sup_{n\ge1}\{\|u-y_n\|\}$. This implies that

$$\begin{aligned}\|S\rho_{n+1}-S\rho_n\|-\|x_{n+1}-x_n\| &\le \|\rho_{n+1}-\rho_n\|-\|x_{n+1}-x_n\|\\ &\le |\alpha_{n+1}-\alpha_n|M_3+|\gamma_{n+1}-\gamma_n|M_2.\end{aligned}$$

From the conditions (a) and (c), we arrive at

$$\limsup_{n\to\infty}(\|S\rho_{n+1}-S\rho_n\|-\|x_{n+1}-x_n\|)\leq 0.$$

Thanks to Lemma 1.2, we obtain that

$$\lim_{n\to\infty}\|S\rho_n-x_n\|=0. \tag{2.2}$$

Notice that

$$\|x_{n+1}-x_n\|=(1-\beta_n)\|S\rho_n-x_n\|.$$

From the condition (b), we see that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.3}$$

Step 3. Show that $x_n - Sx_n \to 0$ as $n\to\infty$.

For each $p\in F$, we have

$$\begin{aligned}\|\rho_n-p\|^2 &= \|\alpha_n u+(1-\alpha_n)y_n-p\|^2\\
&\leq \alpha_n\|u-p\|^2+(1-\alpha_n)\|y_n-p\|^2\\
&= \alpha_n\|u-p\|^2+(1-\alpha_n)\|\gamma_n u_n+(1-\gamma_n)v_n-p\|^2\\
&\leq \alpha_n\|u-p\|^2+(1-\alpha_n)\gamma_n\|T_r(I-rA)x_n-T_r(I-rA)p\|^2\\
&\quad +(1-\alpha_n)(1-\gamma_n)\|T_s(I-sB)x_n-T_s(I-sB)p\|^2\\
&\leq \alpha_n\|u-p\|^2+(1-\alpha_n)\gamma_n\|x_n-p-r(Ax_n-Ap)\|^2\\
&\quad +(1-\alpha_n)(1-\gamma_n)\|x_n-p-s(Bx_n-Bp)\|^2\\
&= \alpha_n\|u-p\|^2+(1-\alpha_n)\gamma_n(\|x_n-p\|^2-2r\langle x_n-p,Ax_n-Ap\rangle+r^2\|Ax_n-Ap\|^2)\\
&\quad +(1-\alpha_n)(1-\gamma_n)(\|x_n-p\|^2-2s\langle x_n-p,Bx_n-Bp\rangle+s^2\|Bx_n-Bp\|^2)\\
&\leq \alpha_n\|u-p\|^2+(1-\alpha_n)\gamma_n[\|x_n-p\|^2-r(2\alpha-r)\|Ax_n-Ap\|^2]\\
&\quad +(1-\alpha_n)(1-\gamma_n)[\|x_n-p\|^2-s(2\beta-s)\|Bx_n-Bp\|^2]\\
&= \alpha_n\|u-p\|^2+(1-\alpha_n)\|x_n-p\|^2-(1-\alpha_n)\gamma_n r(2\alpha-r)\|Ax_n-Ap\|^2\\
&\quad -(1-\alpha_n)(1-\gamma_n)s(2\beta-s)\|Bx_n-Bp\|^2.\end{aligned} \tag{2.4}$$

It follows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\beta_n x_n+(1-\beta_n)S\rho_n-p\|^2\\
&\leq \beta_n\|x_n-p\|^2+(1-\beta_n)\|S\rho_n-p\|^2\\
&\leq \|x_n-p\|^2+\alpha_n\|u-p\|^2-(1-\alpha_n)(1-\beta_n)\gamma_n r(2\alpha-r)\|Ax_n-Ap\|^2\\
&\quad -(1-\alpha_n)(1-\beta_n)(1-\gamma_n)s(2\beta-s)\|Bx_n-Bp\|^2.\end{aligned} \tag{2.5}$$

This implies that

$$\begin{aligned}(1-\alpha_n)(1-\beta_n)\gamma_n r(2\alpha-r)\|Ax_n-Ap\|^2 &\leq \|x_n-p\|-\|x_{n+1}-p\|^2+\alpha_n\|u-p\|^2\\
&\leq (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\alpha_n\|u-p\|^2.\end{aligned}$$

From the conditions (a)–(c) and (2.3), we see that

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0. \tag{2.6}$$

It also follows from (2.5) that

$$(1-\alpha_n)(1-\beta_n)(1-\gamma_n)s(2\beta-s)\|Bx_n-Bp\|^2 \le \|x_n-p\| - \|x_{n+1}-p\|^2+\alpha_n\|u-p\|^2$$
$$\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\alpha_n\|u-p\|^2.$$

From the conditions (a)–(c) and (2.3), we obtain that

$$\lim_{n\to\infty}\|Bx_n-Bp\|=0. \tag{2.7}$$

On the other hand, we have that

$$\begin{aligned}\|u_n-p\|^2 &= \|T_r(I-rA)x_n-T_r(I-rA)p\|^2\\ &\le \langle (I-rA)x_n-(I-rA)p, u_n-p\rangle\\ &= \frac{1}{2}(\|(I-rA)x_n-(I-rA)p\|^2+\|u_n-p\|^2\\ &\quad -\|(I-rA)x_n-(I-rA)p-(u_n-p)\|^2)\\ &\le \frac{1}{2}(\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n-r(Ax_n-Ap)\|^2)\\ &\le \frac{1}{2}[\|x_n-p\|^2+\|u_n-p\|^2-(\|x_n-u_n\|^2-2r\langle x_n-u_n, Ax_n-Ap\rangle\\ &\quad +r^2\|Ax_n-Ap\|^2)].\end{aligned}$$

It follows that

$$\|u_n-p\|^2 \le \|x_n-p\|^2-\|x_n-u_n\|^2+2r\|x_n-u_n\|\|Ax_n-Ap\|. \tag{2.8}$$

Similarly, we can obtain that

$$\|v_n-p\|^2 \le \|x_n-p\|^2-\|x_n-v_n\|^2+2s\|x_n-v_n\|\|Bx_n-Bp\|. \tag{2.9}$$

Note that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\beta_n x_n+(1-\beta_n)S\rho_n-p\|^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|S\rho_n-p\|^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|\alpha_n u+(1-\alpha_n)y_n-p\|^2\\ &\le \beta_n\|x_n-p\|^2+\alpha_n\|u-p\|^2+(1-\beta_n)\|y_n-p\|^2\\ &= \beta_n\|x_n-p\|^2+\alpha_n\|u-p\|^2+(1-\beta_n)\|\gamma_n(u_n-p)+(1-\gamma_n)(v_n-p)\|^2\\ &\le \beta_n\|x_n-p\|^2+\alpha_n\|u-p\|^2+(1-\beta_n)\gamma_n\|u_n-p\|^2+(1-\beta_n)(1-\gamma_n)\|v_n-p\|^2.\end{aligned} \tag{2.10}$$

Substituting (2.8) and (2.9) into (2.10), we see that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \|x_n-p\|^2+\alpha_n\|u-p\|^2-(1-\beta_n)\gamma_n\|x_n-u_n\|^2-(1-\beta_n)(1-\gamma_n)\|x_n-v_n\|^2\\ &\quad +2r\|x_n-u_n\|\|Ax_n-Ap\|+2s\|x_n-v_n\|\|Bx_n-Bp\|.\end{aligned} \tag{2.11}$$

It follows that

$$\begin{aligned}(1-\beta_n)\gamma_n\|x_n-u_n\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|u-p\|^2+2r\|x_n-u_n\|\|Ax_n-Ap\|\\ &\quad +2s\|x_n-v_n\|\|Bx_n-Bp\|\\ &\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\alpha_n\|u-p\|^2+2r\|x_n-u_n\|\|Ax_n-Ap\|\\ &\quad +2s\|x_n-v_n\|\|Bx_n-Bp\|.\end{aligned}$$

From the conditions (a), (c), (2.3), (2.6), and (2.7), we obtain that

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{2.12}$$

It also follows from (2.11) that

$$\begin{aligned}(1-\beta_n)(1-\gamma_n)\|x_n-v_n\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n\|u-p\|^2+2r\|x_n-u_n\|\|Ax_n-Ap\|\\ &\quad +2s\|x_n-v_n\|\|Bx_n-Bp\|\\ &\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\alpha_n\|u-p\|^2+2r\|x_n-u_n\|\|Ax_n-Ap\|\\ &\quad +2s\|x_n-v_n\|\|Bx_n-Bp\|.\end{aligned}$$

Thanks to the conditions (a), (c), (2.3), (2.6), and (2.7), we have that

$$\lim_{n\to\infty}\|x_n-v_n\|=0. \tag{2.13}$$

On the other hand, we have

$$\begin{aligned}\|\rho_n-x_n\| &= \|\alpha_n u+(1-\alpha_n)y_n-x_n\|\\ &\le \alpha_n\|u-x_n\|+(1-\alpha_n)\|y_n-x_n\|\\ &\le \alpha_n\|u-x_n\|+\|\gamma_n u_n+(1-\gamma_n)v_n-x_n\|\\ &\le \alpha_n\|u-x_n\|+\gamma_n\|u_n-x_n\|+(1-\gamma_n)\|v_n-x_n\|.\end{aligned}$$

In view of the condition (a), (2.12), and (2.13), we obtain that

$$\lim_{n\to\infty}\|\rho_n-x_n\|=0. \tag{2.14}$$

It follows that

$$\begin{aligned}\|x_n-Sx_n\| &\le \|x_n-S\rho_n\|+\|S\rho_n-Sx_n\|\\ &\le \|x_n-S\rho_n\|+\|\rho_n-x_n\|.\end{aligned}$$

From (2.2) and (2.13), we see that

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0. \tag{2.15}$$

Step 4. Show that $\lim_{n\to\infty}\langle u-z,\rho_n-z\rangle\le 0$, where $z=P_F u$.

First, we show that

$$\lim_{n\to\infty}\langle u-z,x_n-z\rangle\le 0. \tag{2.16}$$

To show (2.16), we may choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u-z,x_n-z\rangle=\lim_{i\to\infty}\langle u-z,x_{n_i}-z\rangle. \tag{2.17}$$

Since $\{x_{n_i}\}$ is bounded, we can choose a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ that converges weakly to q. We may assume without loss of generality that $x_{n_i}\rightharpoonup q$. Noticing (2.15) and applying Lemma 1.4, we obtain that $q\in F(S)=F(T)$. Next, we define a mapping $R:C\to C$ by

$$Rx=\delta T_r(I-rA)x+(1-\delta)T_s(I-sB)x,\quad \forall x\in C,$$

where $(0,1)\ni\delta=\lim_{n\to\infty}\delta_n$. From Lemma 1.3, we see that R is a nonexpansive mapping with

$$F(R)=F(T_r(I-rA))\cap F(T_s(I-sB))=EP(F_1,A)\cap FP(F_2,B).$$

Note that

$$\|y_n - x_n\| = \|\delta_n u_n + (1-\delta_n)v_n - [\delta_n x_n + (1-\delta_n)x_n]\|$$
$$\leq \delta_n \|u_n - x_n\| + (1-\delta_n)\|v_n - x_n\|.$$

From (2.12) and (2.13), we see that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

On the other hand, we have

$$\|x_n - Rx_n\| \leq \|x_n - y_n\| + \|y_n - Rx_n\|$$
$$= \|x_n - y_n\| + \|\delta_n T_r(I-rA)x_n + (1-\delta_n)T_s(I-sB)x_n - [\delta T_r(I-rA)x_n + (1-\delta)T_s(I-sB)x_n]\|$$
$$\leq \|x_n - y_n\| + |\delta_n - \delta| M_4,$$

where M_4 is an appropriate constant such that $M_4 \geq \sup_{n\geq 1}\{\|T_r(I-rA)x_n\| + \|T_s(I-sB)x_n\|\}$. It follows that $\lim_{n\to\infty}\|Rx_n - x_n\| = 0$. This implies that

$$\lim_{n_i\to\infty} \|Rx_{n_i} - x_{n_i}\| = 0.$$

In view of Lemma 1.4, we obtain that $q \in F(R)$. That is,

$$q \in EP(F_1, A) \cap FP(F_2, B) \cap F(T).$$

It follows from (2.17) that

$$\limsup_{n\to\infty} \langle u - z, x_n - z\rangle = \lim_{i\to\infty} \langle u - z, x_{n_i} - z\rangle = \langle u - z, q - z\rangle \leq 0.$$

Notice that

$$\langle u - z, \rho_n - z\rangle = \langle u - z, \rho_n - x_n\rangle + \langle u - z, x_n - z\rangle$$
$$\leq \|u - z\|\|\rho_n - x_n\| + \langle u - z, x_n - z\rangle.$$

From (2.14), we conclude that

$$\limsup_{n\to\infty} \langle u - z, \rho_n - z\rangle \leq 0. \tag{2.18}$$

Step 5. Show that $x_n \to z$ as $n \to \infty$.

Notice that

$$\|x_{n+1} - z\|^2 = \|\beta_n x_n + (1-\beta_n)S[\alpha_n u + (1-\alpha_n)y_n] - z\|^2$$
$$\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\|S[\alpha_n u + (1-\alpha_n)y_n] - z\|^2$$
$$\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\|\alpha_n u + (1-\alpha_n)y_n - z\|^2$$
$$\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)[(1-\alpha_n)^2\|y_n - z\|^2 + 2\alpha_n\langle u - z, \rho_n - z\rangle]$$
$$\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)[(1-\alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, \rho_n - z\rangle]$$
$$\leq [1 - \alpha_n(1-\beta_n)]\|x_n - z\|^2 + 2\alpha_n(1-\beta_n)\langle u - z, \rho_n - z\rangle.$$

Since $\alpha_n(1-\beta_n) \to 0$, $\sum_{n=1}^{\infty} \alpha_n(1-\beta_n) = \infty$ and $\lim_{n\to\infty} 2\langle u - z, \rho_n - z\rangle \leq 0$, we get the desired conclusion by Lemma 1.5. This completes the proof.

3. APPLICATIONS

First, we consider the following convex feasibility problem (CFP):

$$\text{finding an } x \in \bigcap_{i=1}^{N} C_i,$$

where $N \geq 1$ is an integer and each C_i is assumed to be the set of solutions of an equilibrium problem with the bi-functions F_i, $i = 1,2,\ldots,N$. There is a considerable investigation on CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration [8,11], computer tomography [18], and radiation therapy treatment planning [6]. The following result can be obtained from Theorem 2.1. We, therefore, omit the proof.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $F_1, F_2, \ldots, F_r$ be r bi-functions from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $A_i : C \to H$ be a k_i-inverse-strongly monotone mapping for each $i \in \{1,2,\ldots,r\}$. Assume that $F = \cap_{i=1}^{r} EP(F_i, A_i) \neq \emptyset$. Let $u \in C$, $x_1 \in C$, and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} F_1(u_{n,1},u_1) + \langle A_1 x_n, u_1 - u_{n,1}\rangle + \frac{1}{s_1}\langle u_1 - u_{n,1}, u_{n,1} - x_n\rangle \geq 0, \quad \forall u_1 \in C, \\ F_2(u_{n,2},u_2) + \langle A_2 x_n, u_2 - u_{n,2}\rangle + \frac{1}{s_2}\langle u_2 - u_{n,2}, u_{n,2} - x_n\rangle \geq 0, \quad \forall u_2 \in C, \\ \vdots \\ \vdots \\ F_r(u_{n,r},u_r) + \langle A_r x_n, u_r - u_{n,r}\rangle + \frac{1}{s_r}\langle u_r - u_{n,r}, u_{n,r} - x_n\rangle \geq 0, \quad \forall u_r \in C, \\ y_n = \sum_{i=1}^{r} \gamma_{n,i} u_{n,i}, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)[\alpha_n u + (1-\alpha_n) y_n], \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_{n,i}\}$ are sequences in $(0,1)$, $s_i \in (0, 2k_i)$ for each $i \in \{1,2,\ldots,r\}$. If the above control sequences satisfy the following restrictions
(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(c) $\sum_{i=1}^{r} \gamma_{n,i} = 1$, $\lim_{n\to\infty} \gamma_{n,i} = \gamma_i \in (0,1)$ *for each* $i \in \{1,2,\ldots,r\}$,
then the sequence $\{x_n\}$ converges strongly to $z \in F$, where $z = P_F u$.

Theorem 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H. Let F_1 and F_2 be two bi-functions from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, respectively. Let $T : C \to C$ be a k-strict pseudo-contraction with a fixed point. Define a mapping $S : C \to C$ by $Sx = kx + (1-k)Tx$, $\forall x \in C$. Assume that $F = EP(F_1) \cap EP(F_2) \cap F(T) \neq \emptyset$. Let $u \in C$, $x_1 \in C$, and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} F_1(u_n,u) + \frac{1}{r}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ F_2(v_n,v) + \frac{1}{s}\langle v - v_n, v_n - x_n\rangle \geq 0, \quad \forall v \in C, \\ y_n = \gamma_n u_n + (1-\gamma_n) v_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S[\alpha_n u + (1-\alpha_n) y_n], \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$, $r > 0$, and $s > 0$. If the above control sequences satisfy the following restrictions
(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(c) $\lim_{n\to\infty} \gamma_n = \gamma \in (0,1)$,
then the sequence $\{x_n\}$ converges strongly to $z \in F$, where $z = P_F u$.

Proof. Putting $A = B = 0$, we see that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|, \quad \forall x,y \in C, \alpha > 0$$

and

$$\langle x-y, Bx-By\rangle \geq \beta\|Bx-By\|, \quad \forall x,y \in C, \beta > 0.$$

From Theorem 2.1, we can draw the desired conclusion immediately.

Theorem 3.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and $B : C \to H$ a β-inverse-strongly monotone mapping. Let $T : C \to C$ be a k-strict pseudo-contraction with a fixed point. Define a mapping $S : C \to C$ by $Sx = kx + (1-k)Tx$, $\forall x \in C$. Assume that $F = VI(C,A) \cap VI(C,B) \cap F(T) \neq \emptyset$. Let $u \in C$, $x_1 \in C$, and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \gamma_n P_C(x_n - rAx_n) + (1-\gamma_n)P_C(x_n - sBx_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S[\alpha_n u + (1-\alpha_n)y_n], \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$, $r \in (0,2\alpha)$, and $s \in (0,2\beta)$. If the above control sequences satisfy the following restrictions
(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(c) $\lim_{n\to\infty} \gamma_n = \gamma \in (0,1)$,
then the sequence $\{x_n\}$ converges strongly to $z \in F$, where $z = P_F u$.

Proof. Putting $F_1 \equiv 0$, we see that

$$F_1(u_n, u) + \langle Ax_n, u - u_n\rangle + \frac{1}{r}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C$$

is equivalent to

$$\langle x_n - rAx_n - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C.$$

That is, $u_n = P_C(x_n - rAx_n)$. Similarly, putting $F_2 \equiv 0$, we can obtain that $v_n = P_C(x_n - sBx_n)$. From the proof of Theorem 2.1, we can draw the desired conclusion easily.

Next, we consider another class of nonlinear mappings: strict pseudo-contractions.

Theorem 3.4. *Let C be a nonempty closed convex subset of a Hilbert space H. Let F_1 and F_2 be two bi-functions from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$, respectively. Let $T_A : C \to C$ be a k_α-strict pseudo-contraction and $T_B : C \to H$ a k_β-strict pseudo-contraction. Let $T : C \to C$ be a k-strict pseudo-contraction with a fixed point. Define a mapping $S : C \to C$ by $Sx = kx + (1-k)Tx$, $\forall x \in C$. Assume $F = EP(F_1,(I-T_B)) \cap EP(F_2,(I-T_B)) \cap F(T) \neq \emptyset$. Let $u \in C$, $x_1 \in C$, and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} F_1(u_n, u) + \langle (I-T_A)x_n, u - u_n\rangle + \frac{1}{r}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C, \\ F_2(v_n, v) + \langle (I-T_B)x_n, v - v_n\rangle + \frac{1}{s}\langle v - v_n, v_n - x_n\rangle \geq 0, \quad \forall v \in C, \\ y_n = \gamma_n u_n + (1-\gamma_n)v_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)S[\alpha_n u + (1-\alpha_n)y_n], \quad \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$, $r \in (0,(1-k_\alpha))$, and $s \in (0,(1-k_\beta))$. If the above control sequences satisfy the following restrictions
(a) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,
(c) $\lim_{n\to\infty} \gamma_n = \gamma \in (0,1)$,
then the sequence $\{x_n\}$ will converge strongly to $z \in F$, where $z = P_F u$.

Proof. Putting $A = I - T_A$ and $B = I - T_B$, respectively, we see that A is $\frac{1-k_\alpha}{2}$-inverse-strongly monotone and B is $\frac{1-k_\beta}{2}$-inverse-strongly monotone. The desired result is not hard to derive from the proof of Theorem 2.1.

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Üldistatud tasakaalu ja püsipunkti ülesannete koonduvusteoreemid rakendustega

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On toodud iteratiivne algoritm ühise elemendi leidmiseks üldistatud tasakaalu probleemide lahendite ja otseste pseudokontraktsioonide püsipunktide hulgast. On toodud tugevad koonduvusteoreemid Hilberti ruumides, mis parandavad paljude autorite avaldatud vastavaid tulemusi.