Proceedings of the Estonian Academy of Sciences,
2009, **58**, 3, 146–161
doi: 10.3176/proc.2009.3.02
Available online at www.eap.ee/proceedings

MATHEMATICS

Hypersurfaces with pointwise 1-type Gauss map in Lorentz–Minkowski space

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Received 17 September 2008, accepted 3 November 2008

Abstract. Hypersurfaces of a Lorentz–Minkowski space L^{n+1} with pointwise 1-type Gauss map are characterized. We prove that an oriented hypersurface M_q in L^{n+1} has pointwise 1-type Gauss map of the first kind if and only if M_q has constant mean curvature and conclude that all oriented isoparametric hypersurfaces in L^{n+1} have 1-type Gauss map. Then we classify rational rotation hypersurfaces of L^{n+1} with pointwise 1-type Gauss map and give some examples.

Key words: differential geometry, rotation hypersurface, pointwise Gauss map, finite type, mean curvature.

1. INTRODUCTION

The notion of finite type submanifolds in Euclidean space or pseudo-Euclidean space was introduced by B. Y. Chen in the late 1970s (cf. [5,6]). Since then the theory of submanifolds of finite type has been studied by many geometers and many interesting results have been obtained (see [7] for a report on this subject).

In [9] the notion of finite type was extended to differentiable maps, in particular, to Gauss map of submanifolds. The notion of finite type Gauss map is especially a useful tool in the study of submanifolds (cf. [1–4,9,14,15,19]).

If a submanifold M of a pseudo-Euclidean space $\mathbb{E}^m_s$ has 1-type Gauss map G, then G satisfies $\Delta G = \lambda(G+C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C. However, the Laplacian of the Gauss map of several surfaces and hypersurfaces, such as catenoids and right cones in $\mathbb{E}^3$ [10], generalized catenoids and right n-cones in $\mathbb{E}^{n+1}$ [11], and helicoids of the 1st, 2nd, and 3rd kind, conjugate Enneper's surfaces of the second kind, and B-scrolls in $\mathbb{E}^3_1$ [16] take the form

$$\Delta G = f(G+C) \tag{1.1}$$

for some non-constant function f on M and some constant vector C. A submanifold is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1.1) for some smooth function f on M and some constant vector C. A pointwise 1-type Gauss map is called *proper* if the function f is non-constant. A submanifold with pointwise 1-type Gauss map is said to be of *the first kind* if the vector C in (1.1) is the zero vector. Otherwise, a submanifold with pointwise 1-type Gauss map is said to be of *the second kind*.

In [16], Kim and Yoon gave the complete classification of ruled surfaces in a 3-dimensional Minkowski space with pointwise 1-type Gauss map; in [18] they characterized ruled surfaces of an m-dimensional Minkowski space E^m_1 in terms of the notion of pointwise 1-type Gauss map, and moreover, they studied rotation surfaces of the pseudo-Euclidean space E^4_2 with pointwise 1-type Gauss map in [17]. Recently, in [13], U-H. Ki, D.-S. Kim, Y. H. Kim, and Y.-M. Roh gave a complete classification of rational surfaces of revolution in Minkowski 3-space with pointwise 1-type Gauss map.

In this paper our aim is to study hypersurfaces of a Lorentz–Minkowski space L^{n+1} with pointwise 1-type Gauss map. We first obtain a characterization of hypersurfaces M_q of index q of L^{n+1} with pointwise 1-type Gauss map, that is, we show that an oriented hypersurface M_q of a Lorentz–Minkowski space L^{n+1} has pointwise 1-type Gauss map of the first kind if and only if M_q has constant mean curvature. As a consequence of this, all oriented isoparametric hypersurfaces of L^{n+1} have 1-type Gauss map. Then we classify rational rotation hypersurfaces of L^{n+1} with pointwise 1-type Gauss map which extend the results given in [13] on rational surfaces of revolution in L^3 to the hypersurfaces of L^{n+1}. We also give examples of a rational rotation hypersurface with pointwise 1-type Gauss map of the first and second kind.

2. PRELIMINARIES

Let L^{n+1} denote the $(n+1)$-dimensional Lorentz–Minkowski space, that is, the real vector space $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric $\langle,\rangle = (dx_1)^2 + \cdots + (dx_n)^2 - (dx_{n+1})^2$, where $(x_1,\ldots,x_{n+1})$ are the canonical coordinates in $\mathbb{R}^{n+1}$. A vector x of L^{n+1} is said to be space-like if $\langle x,x\rangle > 0$ or $x = 0$, time-like if $\langle x,x\rangle < 0$, or light-like (or null) if $\langle x,x\rangle = 0$ and $x \neq 0$.

An immersed hypersurface M_q of L^{n+1} with index q ($q = 0,1$) is called space-like (Riemannian) or time-like (Lorentzian) if the induced metric which, as usual, is also denoted by $\langle,\rangle$ on M_q has the index 0 or 1, respectively. The de Sitter n-space $\mathbb{S}^n_1(x_0,c)$ centred at $x_0 \in L^{n+1}$, $c > 0$, is a Lorentzian hypersurface of L^{n+1} defined by

$$\mathbb{S}^n_1(x_0,c) = \{x \in L^{n+1} | \ \langle x-x_0, x-x_0\rangle = c^2\}$$

and the hyperbolic space $\mathbb{H}^n(x_0,-c)$ centred at $x_0 \in L^{n+1}$, $c > 0$, is a space-like hypersurface of L^{n+1} defined by

$$\mathbb{H}^n(x_0,-c) = \{x \in L^{n+1} | \ \langle x-x_0, x-x_0\rangle = -c^2 \text{ and } x_{n+1} - x^0_{n+1} > 0\},$$

where $x_{n+1} - x^0_{n+1}$ is the $(n+1)$-th component of $x - x_0$.

Let Π be a 2-dimensional subspace of L^{n+1} passing through the origin. We will say that Π is non-degenerate if the metric $\langle,\rangle$ restricted to Π is a non-degenerate quadratic form. A curve in L^{n+1} is called space-like, time-like, or light-like if the tangent vector at any point is space-like, time-like, or light-like, respectively.

Here we will define non-degenerate rotation hypersurfaces in L^{n+1} with a time-like, space-like, or light-like axis. For an open interval $I \subset \mathbb{R}$, let $\gamma : I \longrightarrow \Pi$ be a regular smooth curve in a non-degenerate 2-plane Π of L^{n+1} and let ℓ be a line in Π that does not meet the curve γ. A rotation hypersurface M_q with index q in L^{n+1} with a rotation axis ℓ is defined as the orbit of a curve γ under the orthogonal transformations of L^{n+1} with a positive determinant that leaves the rotation axis ℓ fixed (for details see [12]). The curve γ is called a profile curve of the rotation hypersurface. As we consider non-degenerate rotation hypersurfaces, it is sufficient to consider the case that the profile curve is space-like or time-like. The explicit parametrizations for non-degenerate rotation hypersurfaces M_q in L^{n+1} were given in [12] according to the axis ℓ being time-like, space-like, or light-like.

Let $\{\eta_1,\ldots,\eta_{n+1}\}$ be the standard orthonormal basis of L^{n+1}, that is, $\langle \eta_i,\eta_j\rangle = \delta_{ij}$, $\langle \eta_{n+1},\eta_{n+1}\rangle = -1$, $\langle \eta_i,\eta_{n+1}\rangle = 0$, $i,j = 1,2,\ldots,n$. Let $\Theta(u_1,\ldots u_{n-2})$ denote an orthogonal parametrization of the unit sphere $S^{n-2}(1)$ in the Euclidean space E^{n-1} generated by $\{\eta_1,\ldots,\eta_{n-1}\}$:

$$\begin{aligned}\Theta(u_1,\ldots u_{n-2}) = {}& \cos u_1 \eta_1 + \sin u_1 \cos u_2 \eta_2 \\ & + \cdots + \sin u_1 \cdots \sin u_{n-3} \cos u_{n-2} \eta_{n-2} + \sin u_1 \cdots \sin u_{n-3} \sin u_{n-2} \eta_{n-1},\end{aligned} \tag{2.1}$$

where $0 < u_i < \pi$ $(i = 1,\ldots,n-3)$, $0 < u_{n-2} < 2\pi$.

Remark 2.1. When $n = 2$, the term $\Theta(u_1, \ldots, u_{n-2})$ in the following definitions of rotation hypersurfaces is replaced by η_1.

Case 1. ℓ is time-like. In this case the plane Π that contains the line ℓ and a profile curve γ is Lorentzian. Without loss of generality, we may suppose that ℓ is the x_{n+1}-axis and Π is the $x_n x_{n+1}$-plane which is Lorentzian.

Let $\gamma(t) = \varphi(t)\eta_n + \psi(t)\eta_{n+1}$ be a parametrization of γ in the plane Π with $x_n = \varphi(t) > 0$, $t \in I \subset \mathbb{R}$. The curve is space-like if $\varepsilon = \text{sgn}(\varphi'^2 - \psi'^2) = 1$ and time-like if $\varepsilon = \text{sgn}(\varphi'^2 - \psi'^2) = -1$. So a parametrization of a rotation hypersurface $M_{q,T}$ of L^{n+1} with a time-like axis is given by

$$f_T(u_1, \ldots, u_{n-1}, t) = \varphi(t) \sin u_{n-1} \Theta(u_1, \ldots, u_{n-2}) + \varphi(t) \cos u_{n-1}\, \eta_n + \psi(t)\eta_{n+1}, \tag{2.2}$$

where $0 < u_{n-1} < \pi$. The second index in $M_{q,T}$ stands for the time-like axis. The hypersurface $M_{q,T}$ is also called a spherical rotation hypersurface of L^{n+1} as parallels of $M_{q,T}$ are spheres $S^{n-1}(0, \varphi(t))$.

Case 2. ℓ is space-like. In this case the plane Π which contains a profile curve is Lorentzian or Riemannian. So there are rotation hypersurfaces of the first and second kind labelled by M_{q,S_1} and M_{q,S_2} in L^{n+1} with a space-like axis.

Subcase 2.1. The plane Π is Lorentzian. Without losing generality we may suppose that ℓ is the x_n-axis, that is, the vector $\eta_n = (0, 0, \ldots, 0, 1, 0)$ is the direction of the rotation axis, and Π is the $x_n x_{n+1}$-plane. Let $\gamma(t) = \psi(t)\eta_n + \varphi(t)\eta_{n+1}$ be a parametrization of γ in the plane Π with $x_{n+1} = \varphi(t) > 0$, $t \in I \subset \mathbb{R}$. Thus a parametrization of a rotation hypersurface of the first kind M_{q,S_1} of L^{n+1} with a space-like axis is given by

$$f_{S_1}(u_1, \ldots, u_{n-1}, t) = \varphi(t) \sinh u_{n-1} \Theta(u_1, \ldots, u_{n-2}) + \psi(t)\eta_n + \varphi(t) \cosh u_{n-1} \eta_{n+1}, \tag{2.3}$$

$0 < u_{n-1} < \infty$, which is also called a hyperbolic rotation hypersurface of L^{n+1} as parallels of M_{q,S_1} are hyperbolic spaces $H^{n-1}(0, -\varphi(t))$.

Subcase 2.2. The plane Π is Riemannian. We may suppose that ℓ is the x_n-axis and Π is the $x_{n-1}x_n$-plane without loss of generality. Let $\gamma(t) = \varphi(t)\eta_{n-1} + \psi(t)\eta_n$ be a parametrization of γ in the plane Π with $x_{n-1} = \varphi(t) > 0$, $t \in I \subset \mathbb{R}$. In this case the curve γ is space-like. Similarly, a parametrization of a rotation hypersurface of the second kind M_{q,S_2} of L^{n+1} with a space-like axis is given by

$$f_{S_2}(u_1, \ldots, u_{n-1}, t) = \varphi(t) \cosh u_{n-1} \Theta(u_1, \ldots, u_{n-2}) + \psi(t)\eta_n + \varphi(t) \sinh u_{n-1} \eta_{n+1}, \tag{2.4}$$

$-\infty < u_{n-1} < \infty$, which is called a pseudo-spherical rotation hypersurface of L^{n+1} as parallels of M_{q,S_2} are pseudo-spheres $S_1^{n-1}(0, \varphi(t))$ when $n > 2$. (If $n = 2$, then $S_1^1 \equiv H^1$.) Also M_{q,S_2} has index 1, that is, $q = 1$.

Case 3. ℓ is light-like. Let $\{\hat{\eta}_1, \ldots, \hat{\eta}_{n+1}\}$ be a pseudo-Lorentzian basis of L^{n+1}, that is, $< \hat{\eta}_i, \hat{\eta}_j >= \delta_{ij}$, $i, j = 1, \ldots, n-1$, $< \hat{\eta}_i, \hat{\eta}_n >=< \hat{\eta}_i, \hat{\eta}_{n+1} >= 0$, $i = 1, 2, \ldots, n-1$, $< \hat{\eta}_n, \hat{\eta}_{n+1} >= 1$, $< \hat{\eta}_n, \hat{\eta}_n >= 0$, $< \hat{\eta}_{n+1}, \hat{\eta}_{n+1} >= 0$. We can choose $\hat{\eta}_1 = (1, 0, \ldots, 0), \ldots$ $\hat{\eta}_{n-1} = (0, \ldots, 1, 0, 0)$, $\hat{\eta}_n = \frac{1}{\sqrt{2}}(0, \ldots, 0, 1, -1)$, $\hat{\eta}_{n+1} = \frac{1}{\sqrt{2}}(0, \ldots, 0, 1, 1)$. We may suppose that ℓ is the line spanned by the null vector $\hat{\eta}_{n+1}$ and Π is the $x_n x_{n+1}$-plane without loss of generality. Let $\gamma(t) = \sqrt{2}\varphi(t)\hat{\eta}_n + \sqrt{2}\psi(t)\hat{\eta}_{n+1}$ be a parametrization of γ in the plane Π with $x_n = \varphi(t) > 0$, $t \in I \subset \mathbb{R}$. Let $\Theta_1(u_1, \ldots, u_{n-2}), \ldots, \Theta_{n-1}(u_1, \ldots, u_{n-2})$ be the components of the orthogonal parametrization $\Theta(u_1, \ldots, u_{n-2})$ given by (2.1) of the unit sphere $S^{n-2}(1)$ in the basis $\{\hat{\eta}_1, \ldots, \hat{\eta}_{n-1}\}$.

Then a parametrization of a rotation hypersurface $M_{q,L}$ of L^{n+1} with a space-like axis is given by

$$f_L(u_1, \ldots, u_{n-1}, t) = 2\varphi(t)u_{n-1}\Theta(u_1, \ldots, u_{n-2}) + \sqrt{2}\varphi(t)\hat{\eta}_n + \sqrt{2}(\psi(t) - \varphi(t)u_{n-1}^2)\hat{\eta}_{n+1}, \;\; u_{n-1} \neq 0. \tag{2.5}$$

The subgroup of Lorentz group which fixes the direction $\hat{\eta}_{n+1}$ of the light-like axis ℓ can be seen in [12].

Note that in the third case if $\varphi(t) = \varphi_0$ or $\psi(t) = \psi_0$ is a constant, the profile curve is degenerate. However, in the other cases if $\varphi(t) = \varphi_0 > 0$ is a constant and $\psi(t) = t$, the rotation hypersurface $M_{1,T}$ is the Lorentzian cylinder $\mathbb{S}^{n-1}(0,\varphi_0) \times L^1$, M_{0,S_1} is the hyperbolic cylinder $\mathbb{H}^{n-1}(0,-\varphi_0) \times \mathbb{R}$, and M_{1,S_2} is the pseudo-spherical cylinder $\mathbb{S}_1^{n-1}(0,\varphi_0) \times \mathbb{R}$. If $\varphi(t) = t$ and $\psi(t) = \psi_0$ is a constant, then $M_{0,T}$ is a space-like hyperplane of L^{n+1}, and M_{1,S_1} and M_{1,S_2} are time-like hyperplanes of L^{n+1}. Therefore all these are rotation hypersurfaces of L^{n+1} with constant mean curvature.

Let ∇ and ∇' denote the Riemannian connection on M_q and L^{n+1}, respectively. Then, for any vector fields X,Y tangent to M_q, we have the Gauss formula

$$\nabla'_X Y = \nabla_X Y + h(X,Y), \tag{2.6}$$

where h is the second fundamental form which is symmetric in X and Y. For a unit normal vector field ξ, the Weingarten formula is given by

$$\nabla'_X \xi = -A_\xi X, \tag{2.7}$$

where A_ξ is the Weingarten map or the shape operator with respect to ξ. The Weingarten map A_ξ is a self-adjoint endomorphism of TM which cannot be diagonalized in general. It is known that h and A_ξ are related by

$$\langle h(X,Y),\xi\rangle = \langle A_\xi X, Y\rangle. \tag{2.8}$$

The covariant derivative of the second fundamental form h is defined by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^\perp{}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{2.9}$$

where $\nabla^\perp$ denotes the linear connection induced on the normal bundle $T^\perp M$. Then the Codazzi equation is given by

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z). \tag{2.10}$$

Also, from (2.9) we have

$$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_X h)(Z,Y). \tag{2.11}$$

For any normal vector ξ the covariant derivative ∇A_ξ of A_ξ is defined by

$$(\nabla_X A_\xi)Y = \nabla_X(A_\xi Y) - A_\xi(\nabla_X Y). \tag{2.12}$$

Let ξ be a unit normal vector. Since $\nabla^\perp{}_X \xi = 0$, we have by (2.9)

$$\langle(\nabla_X A_\xi)Y, Z\rangle = \langle(\bar{\nabla}_X h)(Y,Z),\xi\rangle. \tag{2.13}$$

Let M_q be a hypersurface with index q in L^{n+1}. The map $G : M^n \to Q^n(\varepsilon_G) \subset L^{n+1}$ which sends each point of M_q to the unit normal vector to M_q at the point is called the Gauss map of the hypersurface M_q, where $\varepsilon_G(= \pm 1)$ denotes the signature of the vector G and $Q^n(\varepsilon_G)$ is an n-dimensional space form given by

$$Q^n(\varepsilon_G) = \begin{cases} \mathbb{S}_1^n(0,1) & \text{in } L^{n+1} \quad \text{if } \varepsilon_G = 1 \\ \mathbb{H}^n(0,-1) & \text{in } L^{n+1} \quad \text{if } \varepsilon_G = -1. \end{cases}$$

Let $e_1,\ldots,e_n$ be an orthonormal local tangent frame on a hypersurface M_q of L^{n+1} with signatures $\varepsilon_i = \langle e_i, e_i\rangle = \mp 1$, and A_G denote the shape operator of M_q in the unit normal direction G. Then the mean curvature H of M_q is defined by

$$H = \frac{1}{n}\varepsilon_G(\mathrm{tr}A_G)G = \frac{1}{n}\sum_{i=1}^{n}\varepsilon_G\varepsilon_i\langle A_G(e_i), e_i\rangle G.$$

A space-like hypersurface of L^{n+1} with vanishing mean curvature is called maximal.

3. HYPERSURFACES WITH POINTWISE 1-TYPE GAUSS MAP

In this section we give a characterization of hypersurfaces of Lorentz–Minkowski space with pointwise 1-type Gauss map.

Lemma 3.1. *Let M_q be a hypersurface with index q in a Lorentz–Minkowski space L^{n+1}. Then we have*

$$\operatorname{trace}(\nabla A_G) = n\nabla\alpha, \tag{3.1}$$

where $\alpha = \sqrt{\varepsilon_G \langle H,H\rangle}$ and $\varepsilon_G = \langle G,G\rangle$.

Proof. Let $e_1,\ldots,e_n$ be a local orthonormal tangent basis on M_q with $\varepsilon_i = \langle e_i,e_i\rangle$, $i=1,\ldots,n$. For any vector X tangent to M_q we have by using (2.9)–(2.11) and (2.13)

$$\begin{aligned}
\langle \operatorname{trace}(\nabla A_G), X\rangle &= \sum_{i=1}^{n} \varepsilon_i \langle (\nabla_{e_i} A_G)e_i, X\rangle \\
&= \sum_{i=1}^{n} \varepsilon_i \left\langle (\bar{\nabla}_{e_i} h)(e_i, X), G\right\rangle \\
&= \sum_{i=1}^{n} \varepsilon_i \left\langle (\bar{\nabla}_{e_i} h)(X, e_i), G\right\rangle \\
&= \sum_{i=1}^{n} \varepsilon_i \left\langle (\bar{\nabla}_{X} h)(e_i, e_i), G\right\rangle \\
&= \sum_{i=1}^{n} \varepsilon_i [\left\langle \nabla^{\perp}_X h(e_i,e_i), G\right\rangle - 2\langle h(\nabla_X e_i, e_i), G\rangle] \\
&= \left\langle \nabla^{\perp}_X \Big(\sum_{i=1}^{n} \varepsilon_i h(e_i,e_i)\Big), G\right\rangle - 2\sum_{i,j=1}^{n} \varepsilon_i \omega_i^j(X)\left\langle h(e_j,e_i), G\right\rangle \\
&= \left\langle n\nabla^{\perp}_X H, G\right\rangle = nX\alpha = n\langle \nabla\alpha, X\rangle
\end{aligned}$$

because $\varepsilon_i\omega_i^j + \varepsilon_j\omega_j^i = 0$, where ω_i^j, $i,j=1,\ldots,n$, are the connection forms associated to $e_1,\ldots,e_n$, and $\nabla\alpha$ is the gradient of the mean curvature. Therefore we obtain (3.1). □

Lemma 3.2. *Let M_q be a hypersurface with index q in a Lorentz–Minkowski space L^{n+1}. Then the Laplacian of the Gauss map G is given as*

$$\Delta G = \varepsilon_G \|A_G\|^2 G + n\nabla\alpha, \tag{3.2}$$

where $\|A_G\|^2 = \operatorname{tr}(A_G A_G)$, $\varepsilon_G = \langle G,G\rangle$, and $\alpha = \sqrt{\varepsilon_G\langle H,H\rangle}$.

Proof. Let C_0 be a fixed vector in L^{n+1}. For any vectors X,Y tangent to M using the Gauss and Weingarten formulas we have

$$YX\left\langle G, C_0\right\rangle = -\left\langle \nabla_Y(A_G(X)) + h(A_G(X),Y), C_0\right\rangle. \tag{3.3}$$

Let $e_1,\ldots,e_n$ be a local orthonormal tangent basis on M_q with $\varepsilon_i = \langle e_i,e_i\rangle$. By using (2.12), (3.3), and Lemma 3.1, we calculate the Laplacian of $\left\langle G, C_0\right\rangle$ as follows:

$$\begin{aligned}
\Delta\langle G,C_0\rangle &= \sum_{i=1}^{n}\varepsilon_i(\nabla_{e_i}e_i - e_ie_i)\langle G,C_0\rangle \\
&= \sum_{i=1}^{n}\varepsilon_i\big\langle -A_G(\nabla_{e_i}e_i),C_0\big\rangle + \sum_{i=1}^{n}\varepsilon_i\big\langle \nabla_{e_i}(A_G(e_i)) + h(A_G(e_i),e_i),C_0\big\rangle \\
&= \left\langle \sum_{i=1}^{n}\varepsilon_i\{\nabla_{e_i}(A_G(e_i)) - A_G(\nabla_{e_i}e_i)\},C_0\right\rangle + \left\langle \sum_{i=1}^{n}\varepsilon_i h(A_G(e_i),e_i),C_0\right\rangle \\
&= \big\langle \mathrm{trace}(\nabla A_G) + \|A_G\|^2 G, C_0\big\rangle
\end{aligned} \tag{3.4}$$

as $\sum_{i=1}^{n}\varepsilon_i h(A_G(e_i),e_i) = \varepsilon_G\|A_G\|^2 G$. Since (3.4) holds for any $C_0 \in L^{n+1}$, the proof is complete. □

Now, from definition (1.1) and equation (3.2) we state the following theorem which characterizes the hypersurfaces of Lorentz–Minkowski spaces with pointwise 1-type Gauss map of the first kind.

Theorem 3.3. *Let M_q be an oriented hypersurface with index q in a Lorentz–Minkowski space L^{n+1}. Then M_q has proper pointwise 1-type Gauss map of the first kind if and only if M_q has constant mean curvature and $\|A_G\|^2$ is non-constant.*

Hence we have

Corollary 3.4. *All oriented isoparametric hypersurfaces of a Lorentz–Minkowski space L^{n+1} have 1-type Gauss map.*

For example, space-like hyperplanes, Lorentzian hyperplanes, hyperbolic spaces $\mathbb{H}^n(0,-c)$, de Sitter spaces $\mathbb{S}^n_1(0,c)$, Lorentzian cylinders $\mathbb{S}^{n-1}(0,c)\times L^1$, hyperbolic cylinders $\mathbb{H}^{n-1}(0,-c)\times\mathbb{R}$, and the pseudo-spherical cylinders $\mathbb{S}^{n-1}_1(0,c)\times\mathbb{R}$ of L^{n+1} have 1-type Gauss map.

From Lemma 3.2 we can also state

Theorem 3.5. *If an oriented hypersurface M_q with index q in a Lorentz–Minkowski space L^{n+1} has proper pointwise 1-type Gauss map of the second kind, then the mean curvature of M is a non-constant function on M_q.*

4. ROTATION HYPERSURFACES WITH POINTWISE 1-TYPE GAUSS MAP OF THE FIRST AND THE SECOND KIND

In this section we obtain a classification of rotation hypersurfaces of L^{n+1} with pointwise 1-type Gauss map of the first and the second kind, and give some examples.

Lemma 4.1. *Let M_q be one of the rotation hypersurfaces $M_{q,T}$, M_{q,S_1}, or M_{1,S_2} of L^{n+1}. If M_q has pointwise 1-type Gauss map in L^{n+1}, then either the Gauss map is harmonic, that is, $\Delta G = 0$ or the function f defined in (1.1) depends only on t and the vector C in (1.1) is parallel to the axis of the rotation of M_q.*

Proof. Let $M_q = M_{q,T}$, which is defined by (2.2). The Gauss map of $M_{q,T}$ is given by

$$G = \frac{1}{\sqrt{\varepsilon(\varphi'^2 - \psi'^2)}}[\psi'(t)(\sin u_{n-1}\Theta + \cos u_{n-1}\eta_n) + \varphi'(t)\eta_{n+1}] \tag{4.1}$$

with $\varepsilon_G = \langle G,G\rangle = -\varepsilon$, where $\varepsilon = \mathrm{sgn}(\varphi'^2 - \psi'^2) = \pm 1$.

The principal curvature of the shape operator A_G of $M_{q,T}$ in the direction G was obtained in [12]. By a direct computation (or following [12]) we have the mean curvature α of $M_{q,T}$ as

$$\alpha = \frac{1}{n\sqrt{\varepsilon(\varphi'^2 - \psi'^2)}}\left(-\frac{(n-1)\psi'}{\varphi} + \frac{\psi'\varphi'' - \varphi'\psi''}{\varphi'^2 - \psi'^2}\right), \tag{4.2}$$

which is the function of t, and the square of the length of the shape operator as

$$\|A_G\|^2 = \frac{\varepsilon}{\varphi'^2 - \psi'^2}\left(\frac{(n-1)\psi'^2}{\varphi^2} + \frac{(\psi'\varphi'' - \varphi'\psi'')^2}{(\varphi'^2 - \psi'^2)^2}\right). \tag{4.3}$$

Since the mean curvature α is the function of t, by a direct computation we obtain the gradient of α as

$$\nabla\alpha = \frac{\alpha'}{\varphi'^2 - \psi'^2}[\varphi'(t)(\sin u_{n-1}\Theta + \cos u_{n-1}\eta_n) + \psi'(t)\eta_{n+1}].$$

Also, by (4.1) we write

$$\nabla\alpha = \frac{\varepsilon\varphi'\alpha'(t)}{\psi'\sqrt{\varepsilon(\varphi'^2 - \psi'^2)}}G - \frac{\alpha'(t)}{\psi'}\eta_{n+1}. \tag{4.4}$$

Using (3.2) and (4.4), the Laplacian of the Gauss map (4.1) becomes

$$\Delta G = \varepsilon\left(\frac{\varphi'\alpha'(t)}{\psi'\sqrt{\varepsilon(\varphi'^2 - \psi'^2)}} - \|A_G\|^2\right)G - \frac{n\alpha'(t)}{\psi'}\eta_{n+1}. \tag{4.5}$$

If M has pointwise 1-type Gauss map, then (1.1) holds for some function f and some vector C. When the Gauss map is not harmonic, equations (1.1), (2.1), (4.1), and (4.5) imply that $C = c\eta_{n+1}$ which is the rotation axis of $M_{q,T}$ for some nonzero constant $c \in \mathbb{R}$, and

$$\varepsilon\left(\frac{\varphi'\alpha'(t)}{\psi'\sqrt{\varepsilon(\varphi'^2 - \psi'^2)}} - \|A_G\|^2\right) = f \quad \text{and} \quad -\frac{n\alpha'(t)}{\psi'} = cf, \tag{4.6}$$

from which the function f is independent of $u_1, \ldots, u_{n-1}$.

In the case $M_q = M_{q,S_1}$ or $M_q = M_{1,S_2}$, we obtain the same result by a similar discussion. □

Theorem 4.2. *There do not exist rotation hypersurfaces M_q in L^{n+1} with a light-like rotation axis and harmonic Gauss map.*

Proof. Without losing generality we may parametrize M_q by (2.5), that is, $M_q = M_{q,L}$. Then the Gauss map $\hat{G}$ of $M_{q,L}$ is given by

$$\hat{G} = \frac{1}{\sqrt{2\hat{\varepsilon}\varphi'\psi'}}[\varphi'(\sqrt{2}u_{n-1}\Theta + \hat{\eta}_n) - (\psi' + \varphi' u_{n-1}^2)\hat{\eta}_{n+1}] \tag{4.7}$$

with $\hat{\varepsilon}_{\hat{G}} = \langle \hat{G}, \hat{G} \rangle = -\hat{\varepsilon}$, where $\hat{\varepsilon} = \operatorname{sgn}(\varphi'\psi') = \pm 1$.

By a direct computation (or see [12]) we have the mean curvature $\hat{\alpha}$ of $M_{q,L}$ as

$$\hat{\alpha} = \frac{1}{n\sqrt{\hat{\varepsilon}\varphi'\psi'}}\left(-\frac{(n-1)\varphi'}{2\varphi} + \frac{\varphi'\psi'' - \psi'\varphi''}{4\varphi'\psi'}\right), \tag{4.8}$$

which is the function of t, and the square of the length of the shape operator $A_{\hat{G}}$ as

$$\|A_{\hat{G}}\|^2 = \frac{\hat{\varepsilon}}{4\varphi'\psi'}\left(\frac{(n-1)\varphi'^2}{\varphi^2} + \frac{(\varphi'\psi'' - \psi'\varphi'')^2}{4\varphi'^2\psi'^2}\right). \tag{4.9}$$

Since the mean curvature $\hat{\alpha}$ is the function of t, we then obtain the gradient of $\hat{\alpha}$ as

$$\nabla\hat{\alpha} = \frac{\sqrt{2}\hat{\alpha}'(t)}{4\varphi'\psi'}[\varphi'(\sqrt{2}u_{n-1}\Theta + \hat{\eta}_n) + (\psi' - \varphi' u_{n-1}^2)\hat{\eta}_{n+1}].$$

Also, by using (4.7), we have

$$\nabla\hat{\alpha} = \frac{\hat{\varepsilon}\hat{\alpha}'(t)}{2\sqrt{\hat{\varepsilon}\varphi'\psi'}}G + \frac{\sqrt{2}\hat{\alpha}'(t)}{2\varphi'}\hat{\eta}_{n+1}. \tag{4.10}$$

Using (3.2) and (4.10), the Laplacian of the Gauss map (4.7) becomes

$$\Delta\hat{G} = \hat{\varepsilon}\left(\frac{n\hat{\alpha}'(t)}{2\sqrt{\hat{\varepsilon}\varphi'\psi'}} - \|A_{\hat{G}}\|^2\right)\hat{G} + \frac{\sqrt{2}n\hat{\alpha}'(t)}{2\varphi'}\hat{\eta}_{n+1}. \tag{4.11}$$

Suppose that the Gauss map is harmonic, that is, $\Delta\hat{G} = 0$. Then, considering (4.7), we have $\|A_{\hat{G}}\| = 0$ from (4.11), which implies $\varphi' = 0$ because of (4.9). This is not possible as the hypersurface is nondegenerate, that is, $\varphi'\psi' \neq 0$. Therefore the Gauss map of $M_{q,L}$ is not harmonic. □

Lemma 4.3. *Let $M_{q,L}$ be a rotation hypersurface of L^{n+1} with a light-like rotation axis parametrized by* (2.5). *If $M_{q,L}$ has pointwise* 1*-type Gauss map in L^{n+1}, then the function f in* (1.1) *depends only on t, and the vector C in* (1.1) *is parallel to the rotation axis.*

Proof. The Gauss map of $M_{q,L}$ and its Laplacian are given by (4.7) and (4.11), respectively. Suppose that $M_{q,L}$ has pointwise 1-type Gauss map in L^{n+1}. By (1.1), (2.1), (4.7), and (4.11) we see that $C = c\hat{\eta}_{n+1}$ which is the rotation axis of $M_{q,L}$ for some nonzero constant $c \in \mathbb{R}$, and

$$\hat{\varepsilon}\left(\frac{n\hat{\alpha}'(t)}{2\sqrt{\hat{\varepsilon}\varphi'\psi'}} - \|A_{\hat{G}}\|^2\right) = f \text{ and } \frac{\sqrt{2}n\hat{\alpha}'(t)}{2\varphi'} = cf, \tag{4.12}$$

from which the function f is independent of $u_1, \ldots, u_{n-1}$. □

Here we give some examples for later use. Let $\varphi(t) = t$, $t > 0$ and $\psi(t) = g(t)$ in the definitions of rotation hypersurfaces $M_{q,T}$, M_{q,S_1}, M_{1,S_2}, and $M_{q,L}$, where $g(t)$ is a differentiable function. In [12], the following results were obtained for the rotation hypersurfaces of L^{n+1} with constant mean curvature:

1) The rotation hypersurface $M_{q,T}$ of L^{n+1} has constant mean curvature α if and only if the function $g(t)$ is given by

$$g(t) = \int^t \frac{a \pm \alpha t^n}{\sqrt{(a \pm \alpha t^n)^2 + \varepsilon t^{2(n-1)}}}dt, \tag{4.13}$$

where a is an arbitrary constant, $\varepsilon = \text{sgn}(1 - g'^2) = \pm 1$, and $q = 0$ for $\varepsilon = 1$ and $q = 1$ for $\varepsilon = -1$.

2) The rotation hypersurface of the first kind M_{q,S_1} of L^{n+1} has constant mean curvature $\bar{\alpha}$ if and only if the function $g(t)$ is given by

$$g(t) = \int^t \frac{a \pm \bar{\alpha} t^n}{\sqrt{(a \pm \bar{\alpha} t^n)^2 - \bar{\varepsilon} t^{2(n-1)}}}dt, \tag{4.14}$$

where a is an arbitrary constant, $\bar{\varepsilon} = \text{sgn}(g'^2 - 1) = \pm 1$, and $q = 0$ for $\bar{\varepsilon} = 1$ and $q = 1$ for $\bar{\varepsilon} = -1$.

3) The Lorentzian rotation hypersurface of the second kind M_{1,S_2} of L^{n+1} has constant mean curvature $\tilde{\alpha}$ if and only if the function $g(t)$ is given by

$$g(t) = \int^t \frac{a \pm \tilde{\alpha} t^n}{\sqrt{t^{2(n-1)} - (a \pm \tilde{\alpha} t^n)^2}} dt, \tag{4.15}$$

where a is an arbitrary constant.

4) The rotation hypersurface $M_{q,L}$ of L^{n+1} has constant mean curvature $\hat{\alpha}$ if and only if the function $g(t)$ is given by

$$g(t) = \int^t \hat{\varepsilon} \frac{t^{2(n-1)}}{(a - 2\hat{\alpha} t^n)^2} dt, \tag{4.16}$$

where a is an arbitrary constant, $\hat{\varepsilon} = \operatorname{sgn}(g') = \pm 1$, and $q = 0$ for $\hat{\varepsilon} = 1$ and $q = 1$ for $\hat{\varepsilon} = -1$.

Example 4.4. The rotation hypersurface $M_{q,T}$ of L^{n+1} defined by (2.2) for the function $g(t)$ given by (4.13) has the Gauss map from (4.1) as

$$G = \frac{a \pm \alpha t^n}{t^{n-1}} (\sin u_{n-1} \Theta + \cos u_{n-1} \eta_n) + \frac{\sqrt{(a \pm \alpha t^n)^2 + \varepsilon t^{2(n-1)}}}{t^{n-1}} \eta_{n+1} \tag{4.17}$$

with $\varepsilon_G = \langle G, G \rangle = -\varepsilon$. Since $M_{q,T}$ has constant mean curvature, we have the Laplacian of the Gauss map by using (3.2) and (4.3) as

$$\Delta G = -\varepsilon \left(n\alpha^2 + \frac{n(n-1)a^2}{t^{2n}} \right) G,$$

which implies that the rotation hypersurface $M_{q,T}$ for the function (4.13) has proper pointwise 1-type Gauss map of the first kind if $a \neq 0$. For instance, when $\alpha = 0$, the generalized catenoids of the first and the third kind have proper pointwise 1-type Gauss map of the first kind. If $a = 0$ and $\alpha \neq 0$, then $M_{q,T}$ has 1-type Gauss map. In this case, $M_{0,T}$ is a part of a hyperbolic n-space $\mathbb{H}^n(c_0 \eta_{n+1}, -1/|\alpha|)$ when $\varepsilon = 1$, and the Lorentzian rotation hypersurface $M_{1,T}$ of L^{n+1} is a part of the de Sitter n-space $\mathbb{S}^n_1(c_0 \eta_{n+1}, 1/|\alpha|)$ when $\varepsilon = -1$ for some $c_0 \in \mathbb{R}$ ([12]).

Example 4.5. The Gauss map of the rotation hypersurface M_{q,S_1} of L^{n+1} defined by (2.3) for the function $g(t)$ given by (4.14) is given by

$$\bar{G} = \frac{a \pm \bar{\alpha} t^n}{t^{n-1}} (\sinh u_{n-1} \Theta + \cosh u_{n-1} \eta_{n+1}) + \frac{\sqrt{(a \pm \alpha t^n)^2 - \bar{\varepsilon} t^{2(n-1)}}}{t^{n-1}} \eta_n \tag{4.18}$$

with $\bar{\varepsilon}_{\bar{G}} = \langle \bar{G}, \bar{G} \rangle = -\bar{\varepsilon}$. By a direct calculation from (3.2) we have the Laplacian of the Gauss map as

$$\Delta \bar{G} = -\bar{\varepsilon} \left(n\bar{\alpha}^2 + \frac{n(n-1)a^2}{t^{2n}} \right) \bar{G},$$

which implies that the rotation hypersurface M_{q,S_1} for the function (4.14) has proper pointwise 1-type Gauss map of the first kind if $a \neq 0$. For instance, when $\bar{\alpha} = 0$, the generalized catenoids of the second and the fourth kind have proper pointwise 1-type Gauss map of the first kind. If $a = 0$ and $\bar{\alpha} \neq 0$, then M_{q,S_1} has 1-type Gauss map. In this case, M_{0,S_1} is a part of a hyperbolic n-space $\mathbb{H}^n(c_0 \eta_{n+1}, -1/|\bar{\alpha}|)$ when $\bar{\varepsilon} = 1$, and the Lorentzian rotation hypersurface M_{1,S_1} of L^{n+1} is a part of the de Sitter n-space $\mathbb{S}^n_1(c_0 \eta_{n+1}, 1/|\bar{\alpha}|)$ when $\bar{\varepsilon} = -1$ for some $c_0 \in \mathbb{R}$ ([12]).

Example 4.6. Now we consider the rotation hypersurface M_{1,S_2} of L^{n+1} defined by (2.4) for the function $g(t)$ given by (4.15). Then the Gauss map $\tilde{G}$ of M_{1,S_2} is obtained as

$$\tilde{G} = \frac{a \pm \tilde{\alpha} t^n}{t^{n-1}} (\cosh u_{n-1} \Theta + \sinh u_{n-1} \eta_{n+1}) - \frac{\sqrt{t^{2(n-1)} - (a \pm \tilde{\alpha} t^n)^2}}{t^{n-1}} \eta_n \tag{4.19}$$

with $\tilde{\varepsilon}_{\tilde{G}} = \langle \tilde{G}, \tilde{G} \rangle = 1$. By a direct calculation from (3.2) we have the Laplacian of the Gauss map as

$$\Delta\tilde{G} = \left(n\tilde{\alpha}^2 + \frac{n(n-1)a^2}{t^{2n}} \right) \tilde{G},$$

which implies that the rotation hypersurface M_{1,S_2} for the function (4.15) has proper pointwise 1-type Gauss map of the first kind if $a \neq 0$. For instance, when $\tilde{\alpha} = 0$, the generalized catenoids of the fifth kind have proper pointwise 1-type Gauss map of the first kind. If $a = 0$ and $\tilde{\alpha} \neq 0$, then M_{1,S_2} has 1-type Gauss map, and it is a part of the de Sitter n-space $\mathbb{S}^n_1(c_0\eta_{n+1}, 1/|\tilde{\alpha}|)$ for some $c_0 \in \mathbb{R}$ ([12]).

Example 4.7. The rotation hypersurface $M_{q,L}$ of L^{n+1} defined by (2.5) for the function $g(t)$ given by (4.16) has the Gauss map as

$$\hat{G} = \frac{1}{\sqrt{2}t^{n-1}(a-\hat{\alpha}t^n)} \{ (a-\hat{\alpha}t^n)^2(\sqrt{2}u_{n-1}\Theta + \hat{\eta}_n) - [\hat{\varepsilon}t^{2(n-1)} + (a-\hat{\alpha}t^n)^2 u_{n-1}^2]\hat{\eta}_{n+1} \} \tag{4.20}$$

with $a - \hat{\alpha}t^n > 0$ and $\varepsilon_{\hat{G}} = \langle \hat{G}, \hat{G} \rangle = -\hat{\varepsilon}$, where $\hat{\eta}_n$ are $\hat{\eta}_{n+1}$ vectors in the pseudo-orthonormal basis given in the definition of $M_{q,L}$. By a direct calculation from (3.2) we have the Laplacian of the Gauss map as

$$\Delta\hat{G} = -\hat{\varepsilon} \left(n\hat{\alpha}^2 + \frac{n(n-1)a^2}{4t^{2n}} \right) \hat{G},$$

which implies that the rotation hypersurface $M_{q,L}$ for the function (4.16) has proper pointwise 1-type Gauss map of the first kind if $a \neq 0$. For instance, when $\hat{\alpha} = 0$, the Enneper's hypersurfaces of the second and the third kind [12] have proper pointwise 1-type Gauss map of the first kind. If $a = 0$ and $\hat{\alpha} \neq 0$, then $M_{q,L}$ has 1-type Gauss map. In this case, $M_{0,L}$ is a part of a hyperbolic n-space $\mathbb{H}^n(c_0\hat{\eta}_{n+1}, -1/|\hat{\alpha}|)$ when $\hat{\varepsilon} = 1$, and the Lorentzian rotation hypersurface $M_{1,L}$ of L^{n+1} is a part of the de Sitter n-space $\mathbb{S}^n_1(c_0\hat{\eta}_{n+1}, 1/|\hat{\alpha}|)$ when $\hat{\varepsilon} = -1$ for some $c_0 \in \mathbb{R}$ ([12]).

Example 4.8. (Spherical n-cone) Consider the rotation hypersurface $M_{q,T}$ of L^{n+1} parametrized by (2.2) for the functions $\varphi(t) = t, (t > 0)$ and $\psi(t) = at,\ a > 0$. It is a right n-cone $C_{a,T}$ with a time-like rotation axis based on a sphere $\mathbb{S}^{n-1}$, which is space-like if $0 < a < 1$ and time-like if $|a| > 1$. The Gauss map G of $C_{a,T}$ and its Laplacian ΔG are, respectively, given by

$$G = \frac{1}{\sqrt{\varepsilon(1-a^2)}} [a(\sin u_{n-1}\Theta + \cos u_{n-1}\eta_n) + \eta_{n+1}]$$

and

$$\Delta G = \frac{n-1}{t^2} \left(G - \frac{1}{\sqrt{\varepsilon(1-a^2)}} \eta_{n+1} \right),$$

where $\varepsilon = \operatorname{sgn}(1-a^2)$. Therefore the spherical n-cone $C_{a,T}$ has proper pointwise 1-type Gauss map of the second kind.

Example 4.9. (Hyperbolic n-cone) Now we consider the rotation hypersurface M_{q,S_1} of L^{n+1} parametrized by (2.3) for the functions $\varphi(t) = t, (t > 0)$ and $\psi(t) = at,\ a > 0$. It is a hyperbolic n-cone C_{a,S_1} of the first kind with a space-like rotation axis based on a hyperbolic space $\mathbb{H}^{n-1}$, which is space-like if $|a| > 1$ and time-like if $0 < a < 1$. The Gauss map $\bar{G}$ of C_{a,S_1} and its Laplacian $\Delta\bar{G}$ are, respectively, given by

$$\bar{G} = \frac{1}{\sqrt{\bar{\varepsilon}(a^2-1)}} [a(\sinh u_{n-1}\Theta + \cosh u_{n-1}\eta_{n+1}) + \eta_n]$$

and

$$\Delta\bar{G} = -\frac{n-1}{t^2} \left(\bar{G} - \frac{1}{\sqrt{\bar{\varepsilon}(a^2-1)}} \eta_n \right),$$

where $\bar{\varepsilon} = \mathrm{sgn}(a^2 - 1)$. Therefore the hyperbolic n-cone C_{a,S_1} of the first kind has proper pointwise 1-type Gauss map of the second kind.

Example 4.10. (Pseudo-spherical n-cone) The rotation hypersurface M_{1,S_2} of L^{n+1} parametrized by (2.4) for the functions $\varphi(t) = t, (t > 0)$ and $\psi(t) = at$, $a > 0$ is a hyperbolic n-cone C_{a,S_2} of the second kind with a space-like rotation axis. It is a time-like (Lorentzian) n-cone based on a pseudo-sphere $\mathbb{S}_1^{n-1}$ which has the Gauss map $\tilde{G}$ as

$$\tilde{G} = \frac{1}{\sqrt{\bar{\varepsilon}(a^2+1)}}[a(\cosh u_{n-1}\Theta + \sinh u_{n-1}\eta_{n+1}) - \eta_n]$$

and the Laplacian $\Delta\tilde{G}$ of Gauss map $\tilde{G}$ is given by

$$\Delta\tilde{G} = \frac{n-1}{t^2}\left(\bar{G} + \frac{1}{\sqrt{\bar{\varepsilon}(a^2+1)}}\eta_n\right),$$

where $\bar{\varepsilon} = \mathrm{sgn}(a^2 - 1)$. Therefore the pseudo-spherical n-cone C_{a,S_2} of the second kind has proper pointwise 1-type Gauss map of the second kind.

The notion of rotation surfaces of polynomial and rational kinds was introduced by Chen and Ishikawa in [8]. A rotation hypersurface in L^{n+1} is said to be of *polynomial kind* if the functions $\varphi(t)$ and $\psi(t)$ in the parametrization of the rotation hypersurfaces given in the first section are polynomials, and it is said to be of *rational kind* if $\varphi(t)$ and $\psi(t)$ are rational functions. A rotation hypersurface of rational kind is simply called rational rotation hypersurface. Without loss of generality we consider rotation hypersurfaces $M_{q,T}$, M_{q,S_1}, or M_{1,S_2} in L^{n+1} given by (2.2), (2.3), and (2.4), respectively, for $\varphi(t) = t$, $t > 0$ and $\psi(t) = g(t)$, where $g(t)$ is a function of class C^3.

By the following theorem we classify rational rotation hypersurfaces of L^{n+1} in terms of pointwise 1-type Gauss map of the first kind.

Theorem 4.11.

(1) *A rational rotation hypersurface $M_{q,T}$ of L^{n+1} parametrized by* (2.2) *has pointwise* 1*-type Gauss map of the first kind if and only if it is an open portion of a space-like hyperplane or a Lorentzian cylinder $\mathbb{S}^{n-1} \times L^1$ of L^{n+1}.*

(2) *A rational rotation hypersurface M_{q,S_1} of L^{n+1} parametrized by* (2.3) *has pointwise* 1*-type Gauss map of the first kind if and only if it is an open portion of a time-like hyperplane or a hyperbolic cylinder $\mathbb{H}^{n-1} \times \mathbb{R}$ of L^{n+1}.*

(3) *A rational rotation hypersurface M_{q,S_2} of L^{n+1} parametrized by* (2.4) *has pointwise* 1*-type Gauss map of the first kind if and only if it is an open portion of a time-like hyperplane or a pseudo-spherical cylinder $\mathbb{S}_1^{n-1} \times \mathbb{R}$ of L^{n+1}.*

(4) *A rational rotation hypersurface $M_{q,L}$ of L^{n+1} parametrized by* (2.5) *has pointwise* 1*-type Gauss map of the first kind if and only if it is an open portion of hyperbolic n-space $\mathbb{H}^n$, de Sitter n-space $\mathbb{S}_1^n$ or Enneper's hypersurface of the second kind or the third kind.*

Moreover, the Enneper's hypersurfaces of the second kind and the third kind of L^{n+1} are the only polynomial rotation hypersurfaces of L^{n+1} with proper pointwise 1*-type Gauss map of the first kind.*

Proof. In the parametrization (2.2) of $M_{q,T}$, if φ is a constant, the hypersurface $M_{q,T}$ is an open portion of a Lorentzian cylinder $\mathbb{S}^{n-1} \times L^1$ of L^{n+1}. If φ is not a constant, we put $\varphi = t$, $t > 0$ and $\psi(t) = g(t)$ in the parametrization (2.2) of $M_{q,T}$. In [12] it was shown that $M_{q,T}$ has constant mean curvature α if and only if the function $g(t)$ is given by (4.13). Now, if $a = \alpha = 0$ in (4.13), then $g(t)$ is a constant. In this case, the hypersurface $M_{q,T}$ is an open portion of a space-like hyperplane.

If $a \neq 0$ and $\alpha = 0$, that is, $M_{q,T}$ is the generalized catenoid of the first or the third kind ([12]), then (4.13) implies that the function $g(t)$ can be expressed in terms of elliptic functions and $g(t)$ is not a rational function of t.

If $a = 0$ and $\alpha \neq 0$, then from (4.13), we get $g(t) = \alpha^{-1}\sqrt{\alpha^2 t^2 + \varepsilon} + c$, which is not rational, where c is an arbitrary constant and $t > 1/|\alpha|$ when $\varepsilon = -1$. Therefore $M_{q,T}$ is not rational kind. In this case, the hypersurface $M_{q,T}$ is an open portion of a hyperbolic n-space $\mathbb{H}^{n-1}$ when $\varepsilon = 1$ or an open portion of a de Sitter n-space $\mathbb{S}_1^{n-1}$ when $\varepsilon = -1$.

If $a\alpha \neq 0$, then $g(t)$ given by (4.13) cannot be rational even if $n = 2$. If it were rational, its derivative would be rational, which contradicts the integrand in (4.13). The converse is trivial.

Parts 2 and 3 can similarly be proved by using (2.3), (2.4), (4.14), and (4.15).

For the proof of part 4, we put $\varphi = t$, $t > 0$ and $\psi(t) = g(t)$ in the parametrization (2.5) of $M_{q,L}$. In [12], it was proved that $M_{q,L}$ has constant mean curvature $\hat{\alpha}$ if and only if the function $g(t)$ is given by (4.16). Now, if $a = 0$ and $\hat{\alpha} \neq 0$ in (4.16), then we obtain $g(t) = c - \frac{\hat{\varepsilon}}{4\hat{\alpha}^2 t}$ which is a rational function, and $M_{q,L}$ is an open part of a hyperbolic n-space $\mathbb{H}^n$ when $q = 0$ ($\hat{\varepsilon} = 1$) and $M_{q,L}$ is an open part of a de Sitter n-space $\mathbb{S}_1^n$ when $q = 1$ ($\hat{\varepsilon} = -1$).

If $a \neq 0$ and $\hat{\alpha} = 0$, then we have $g(t) = \frac{\hat{\varepsilon} t^{2n-1}}{a^2(2n-1)} + c$ which is a polynomial. In this case, $M_{q,L}$ is an open portion of Enneper's hypersurface ([12]) of the second or the third kind according to $\hat{\varepsilon} = 1$ or $\hat{\varepsilon} = -1$. From Example 4.7 it is seen that Enneper's hypersurfaces are the only polynomial (rational) rotation hypersurfaces of L^{n+1} with proper pointwise 1-type Gauss map of the first kind.

If $a\hat{\alpha} \neq 0$, then the function $g(t)$ given by (4.16) is not rational for $n \geq 2$ because the integration of $t^{2(n-1)}/(a - 2\hat{\alpha}t^n)^2$ contains at least one term involving a logarithmic or arctangent function. The converse of part 4 follows from Corollary 3.4 and Example 4.7. □

Corollary 4.12. *The rotation hypersurface $M_{q,L}$ of L^{n+1} parametrized by* (2.5) *for the function $g(t) = c - \frac{\hat{\varepsilon}}{4\hat{\alpha}^2 t}$ is the only non-polynomial rational rotation hypersurface of L^{n+1} with pointwise* 1*-type Gauss map.*

The proof follows from the proof of Theorem 4.11 and Example 4.7.

Theorem 4.13. *Let M_q be one of the rotation hypersurfaces $M_{q,T}$, M_{q,S_1}, or M_{1,S_2} in L^{n+1} parametrized by* (2.2), (2.3), *and* (2.4), *respectively. If M_q is a polynomial kind rotation hypersurface, then it has proper pointwise* 1*-type Gauss map of the second kind if and only if it is an open portion of a spherical n-cone, hyperbolic n-cone, or pseudo-spherical n-cone.*

Proof. Let $M_q = M_{q,T}$. In the parametrization (2.2) of $M_{q,T}$ we take $\varphi(t) = t, t > 0$ and $\psi(t) = g(t)$, where $g(t)$ is a polynomial. Then we have the Gauss map G from (4.1) as

$$G = \frac{1}{\sqrt{\varepsilon(1-g'^2)}}[g'(t)(\sin u_{n-1}\Theta + \cos u_{n-1}\eta_n) + \eta_{n+1}] \tag{4.21}$$

with $\varepsilon_G = \langle G, G\rangle = -\varepsilon$, where $\varepsilon = \operatorname{sgn}(1-g'^2) = \pm 1$ and $|g'| \neq 1$. Also, from (4.5) the Laplacian of the Gauss map G is given by

$$\Delta G = \varepsilon\left(\frac{n\alpha'}{g'\sqrt{\varepsilon(1-g'^2)}} - \|A_G\|^2\right)G - \frac{n\alpha'}{g'}\eta_{n+1}, \tag{4.22}$$

where $\|A_G\|^2$ is given by (4.3) for $\varphi(t) = t$ and $\psi(t) = g(t)$ and the derivative of α from (4.2) is evaluated as

$$\alpha'(t) = \frac{1}{n\sqrt{\varepsilon(1-g'^2)}}\left(\frac{(n-1)g'}{t^2} - \frac{(n-1)g''}{t(1-g'^2)} - \frac{g'''}{1-g'^2} - \frac{3g'g''^2}{(1-g'^2)^2}\right). \tag{4.23}$$

Suppose that M has pointwise 1-type Gauss map of the second kind. Then, by definition, the vector C in (1.1) is nonzero and by Lemma 4.1 $C = c\eta_{n+1}$ for some nonzero constant c. Thus, (1.1) and (4.22) imply that

$$\varepsilon\left(\frac{n\alpha'}{g'\sqrt{\varepsilon(1-g'^2)}} - \|A_G\|^2\right) = f \quad \text{and} \quad -\frac{n\alpha'}{g'} = cf.$$

Eliminating f in the above equations and using (4.3) and (4.23), we obtain

$$P(t) = c\sqrt{\varepsilon(1-g'^2)}\,Q(t), \tag{4.24}$$

where

$$P(t) = ((n-1)g'' + tg''')(1-g'^2)^2 t + 3g'g''(1-g'^2)t^2 - (n-1)g'(1-g'^2)^3,$$
$$Q(t) = (n-1)g'(1-g'^2)^3 - (n-1)g''(1-g'^2)t - g'''(1-g'^2)t^2 - 4g'g''^2t^2.$$

If $\deg g(t) \geq 2$, then $\deg P(t) = \deg Q(t) \geq 7$, which is a contradiction. Consequently, $\deg g(t) = 1$. That is, $g'(t) = a$ for some nonzero constant a with $|a| \neq 1$. Hence, we get $c = -1/\sqrt{\varepsilon(1-a^2)}$. Therefore, the rotation hypersurface $M_{q,T}$ with the parametrization (2.2) for $\varphi(t) = t$, $t > 0$ and $\psi(t) = at + b$ is an open portion of a spherical n-cone. The proof of the converse for $M_q = M_{q,T}$ follows from Example 4.8.

By a similar discussion as above it can be shown that if $M_q = M_{q,S_1}$ or $M_q = M_{q,S_2}$, then it is an open portion of a hyperbolic n-cone or an open portion of a pseudo-spherical n-cone, respectively. □

Theorem 4.14. *There do not exist rational rotation hypersurfaces $M_{q,T}$, M_{q,S_1}, or M_{1,S_2} in L^{n+1}, except polynomial hypersurfaces, with pointwise 1-type Gauss map of the second kind.*

Proof. Let $M_q = M_{q,T}$. Assume that $M_{q,T}$ is a rational rotation hypersurface in L^{n+1}, except polynomial hypersurface, with pointwise 1-type Gauss map of the second kind. In the parametrization (2.2) of $M_{q,T}$, we take $\varphi(t) = t, t > 0$ and $\psi(t) = g(t)$, where $g(t)$ is a rational function. The derivatives of $g(t)$ are also rational functions in t. We may put $g'(t) = r(t)/q(t)$, where $r(t)$ and $q(t)$ are relative prime polynomials. Let $\deg q(t) = k$.

From (4.24) we know that $\sqrt{\varepsilon(1-g'^2)}$ is also a rational function. Hence there exists a polynomial $p(t)$ satisfying $q^2(t) - r^2(t) = \varepsilon p^2(t)$, where $r(t)$, $q(t)$, and $p(t)$ are relatively prime. Put

$$P_1(t) = g''(1-g'^2)^2 t, \quad P_2(t) = g'''(1-g'^2)^2 t^2,$$
$$P_3(t) = g'g''(1-g'^2)t^2, \quad P_4(t) = g'(1-g'^2)^3,$$
$$Q_1(t) = g''(1-g'^2)t, \quad Q_2(t) = g'''(1-g'^2)t^2,$$
$$Q_3(t) = g'g''^2t^2, \quad Q_4(t) = P_4(t).$$

Then these functions are also rational.

Suppose that $k \geq 1$. Then, for each $i = 1,\ldots,4$, we see that $q^7(t)P_i(t)$ is a polynomial. Similarly, we see that for each $i = 1,\ldots,3$, $q^6(t)Q_i(t)$ is a polynomial. However, we have

$$q^6(t)Q_4(t) = \frac{\varepsilon r(t)p^6(t)}{q(t)}.$$

As (4.24) gives

$$P(t) = c\frac{p(t)}{q(t)}Q(t), \tag{4.25}$$

it follows that $q^6(t)Q_4(t)$ is a polynomial. This is a contradiction because $r(t)$, $q(t)$, and $p(t)$ are relatively prime. Therefore $g'(t)$ is not rational, so is $g(t)$. Hence $k = 0$, that is, $g(t)$ is a polynomial, and by Theorem 4.13 $M_q = M_{q,T}$ is nothing but a spherical n-cone.

By a similar discussion, when $M_q = M_{q,S_1}$ or $M_q = M_{q,S_2}$, we have the same result. □

Theorem 4.15. *There do not exist rational rotation hypersurfaces $M_{q,L}$ in L^{n+1} with a light-like axis and pointwise* 1*-type Gauss map of the second kind.*

Proof. Suppose that $M_{q,L}$ given by (2.5) is a rational rotation hypersurface with pointwise 1-type Gauss map of the second kind. Then we put $\varphi(t) = t$, $t > 0$ and $\psi(t) = g(t)$ in (2.5), where $g(t)$ is a rational function. From (4.7) and (4.11) the Gauss map $\hat{G}$ of $M_{q,L}$ and its Laplacian $\Delta\hat{G}$ are, respectively, given by

$$\hat{G} = \frac{1}{\sqrt{2\hat{\varepsilon}g'}}\left(\sqrt{2}u_{n-1}\Theta + \hat{\eta}_n - (g' + u_{n-1}^2)\hat{\eta}_{n+1}\right) \tag{4.26}$$

and

$$\Delta\hat{G} = \hat{\varepsilon}\left(\frac{n\hat{\alpha}'(t)}{2\sqrt{\hat{\varepsilon}g'}} - \|A_{\hat{G}}\|^2\right)\hat{G} + \frac{\sqrt{2}n\hat{\alpha}'(t)}{2}\hat{\eta}_{n+1}, \tag{4.27}$$

where $\hat{\varepsilon} = \operatorname{sgn}(g') = \pm 1$,

$$\hat{\alpha}'(t) = \frac{1}{2n\sqrt{\hat{\varepsilon}g'}}\left(\frac{n-1}{t^2} + \frac{(n-1)g''}{2tg'} + \frac{g'''}{2g'} - \frac{3g''^2}{4g'}\right) \tag{4.28}$$

from (4.8), and

$$\|A_{\hat{G}}\|^2 = \frac{\hat{\varepsilon}}{4g'}\left(\frac{n-1}{t^2} + \frac{g''^2}{4g'^2}\right). \tag{4.29}$$

Since $M_{q,L}$ has pointwise 1-type Gauss map of the second kind, by Lemma 4.3 the vector C in the definition (1.1) is parallel to $\hat{\eta}_{n+1}$, that is, $C = c\hat{\eta}_{n+1}$, and (1.1) and (4.27) imply that

$$\hat{\varepsilon}\left(\frac{n\hat{\alpha}'(t)}{2\sqrt{\hat{\varepsilon}g'}} - \|A_{\hat{G}}\|^2\right) = f \quad \text{and} \quad \frac{\sqrt{2}n\hat{\alpha}'(t)}{2} = cf.$$

Eliminating f in the above equations, and using (4.28) and (4.29), we obtain

$$P(t) = c\sqrt{2\hat{\varepsilon}g'}\,Q(t), \tag{4.30}$$

where

$$P(t) = 4g'^3 + 2g'^2g'''t^2 - 3g'g''^2t^2 + 2(n-1)g'^2g''t,$$
$$Q(t) = g'g'''t^2 - 2g''^2t^2 + (n-1)g'g''t,$$

which are rational functions as $g(t)$ is rational. The function $\sqrt{\hat{\varepsilon}g'}$ in (4.30) is also a rational function. Then we may put $g' = \hat{\varepsilon}r^2(t)/q^2(t)$, where $r(t)$ and $q(t)$ are relatively prime polynomials. Taking derivative, we have

$$g''(t) = \frac{\hat{\varepsilon}R_1(t)}{q^3} \quad \text{and} \quad g'''(t) = \frac{\hat{\varepsilon}R_2(t)}{q^4},$$

where

$$R_1(t) = 2r(qr' - rq'),$$
$$R_2(t) = 2(q^2r'^2 + q^2rr'' - 4rqr'q' - r^2qq'' + 3r^2q'^2),$$

which are polynomials in t. Hence,

$$P(t) = \frac{\hat{\varepsilon}r^2\bar{P}(t)}{q^8} \quad \text{and} \quad Q(t) = \frac{\bar{Q}(t)}{q^6},$$

where

$$\bar{P}(t) = r^4q^2 + 2t^2r^2R_2(t) - 3t^2R_1^2(t) + 2(n-1)tr^2qR_1(t),$$
$$\bar{Q}(t) = t^2r^2R_2(t) - 2t^2R_1^2(t) + (n-1)tr^2qR_1(t).$$

Therefore equation (4.30) becomes

$$r(t)\bar{P}(t) = c\hat{\varepsilon}\sqrt{2}q(t)\bar{Q}(t). \tag{4.31}$$

Let $\deg r(t) = m$ and $\deg q(t) = k$. We may write $r(t) = \sum_{s=0}^{m} a_s t^s$ and $q(t) = \sum_{s=0}^{k} b_s t^s$ such that $a_m \neq 0$ and $b_k \neq 0$. Then, by a straightforward computation we obtain

$$R_1(t) = 2(m-k)a_m^2 b_k t^{2m+k-1} + \cdots + a_0(a_1b_0 - a_0b_1) \tag{4.32}$$

and

$$R_2(t) = 2(m-k)(2m-2k-1)a_m^2 b_k^2 t^{2m+2k-2} + \cdots + d_0, \tag{4.33}$$

where $d_0 = 2(a_1^2b_0^2 + 3a_0^2b_1^2 + 2b_0^2a_0a_2 - 2a_0^2b_0b_2 - 4a_0b_0a_1b_1)$. Using (4.32) and (4.33), we get $\deg\bar{P}(t) = 4m+2k$ and $\deg\bar{Q}(t) = 4m+2k$ if $m \neq k$, and $\deg\bar{Q}(t) \leq 4m+2k-1$ if $m = k$.

Now, if $m \neq k$, then $\deg(r(t)\bar{P}(t)) = 5m+2k$ and $\deg(q(t)\bar{Q}(t)) = 4m+3k$. Hence, by comparing the degree of the polynomials $r(t)\bar{P}(t)$ and $q(t)\bar{Q}(t)$, from (4.31) we have $5m+2k = 4m+3k$, which implies that $m = k$, which is a contradiction. If $m = k$, then $\deg(r(t)\bar{P}(t)) = 7m$ and $\deg(q(t)\bar{Q}(t)) \leq 7m-1$, which is also a contradiction because of (4.31). Therefore $\sqrt{2\hat{\varepsilon}g'}$ is not a rational function, and so is $g(t)$. □

Corollary 4.16. *There do not exist polynomial rotation hypersurfaces $M_{q,L}$ in L^{n+1} with a light-like axis and pointwise 1-type Gauss map of the second kind.*

Considering Theorem 4.13, Theorem 4.14, and Theorem 4.15, we have the following classification theorem for rational rotation hypersurfaces of L^{n+1} with pointwise 1-type Gauss map of the second kind.

Theorem 4.17. *Let M be a rational rotation hypersurface of L^{n+1}. Then M has pointwise 1-type Gauss map of the second kind in L^{n+1} if and only if it is an open portion of a spherical n-cone, hyperbolic n-cone, or pseudo-spherical n-cone.*

REFERENCES

1. Baikoussis, C. Ruled submanifolds with finite type Gauss map. *J. Geom.*, 1994, **49**, 42–45.
2. Baikoussis, C. and Blair, D. E. On the Gauss map of ruled surfaces. *Glasgow Math. J.*, 1992, **34**, 355–359.
3. Baikoussis, C. and Verstraelen, L. The Chen-type of the spiral surfaces. *Results in Math.*, 1995, **28**, 214–223.
4. Baikoussis, C., Chen, B. Y., and Verstraelen, L. Ruled surfaces and tubes with finite type Gauss map. *Tokyo J. Math.*, 1993, **16**, 341–348.
5. Chen, B. Y. *Total Mean Curvature and Submanifolds of Finite Type*. World Scientific, Singapore-New Jersey-London, 1984.
6. Chen, B. Y. Finite type submanifolds in pseudo-Euclidean spaces and applications. *Kodai Math. J.*, 1985, **8**, 358–374.
7. Chen, B. Y. A report on submanifolds of finite type. *Soochow J. Math.*, 1996, **22**, 117–337.
8. Chen, B. Y. and Ishikawa, S. On classification of some surfaces of revolution of finite type. *Tsukuba J. Math.*, 1993, **17**, 287–298.
9. Chen, B. Y. and Piccinni, P. Submanifolds with finite type Gauss map. *Bull. Austral. Math. Soc.*, 1987, **35**, 161–186.
10. Chen, B. Y., Choi, M., and Kim, Y. H. Surfaces of revolution with pointwise 1-type Gauss map. *J. Korean Math.*, 2005, **42**, 447–455.
11. Dursun, U. Hypersurfaces with pointwise 1-type Gauss map. *Taiwanese J. Math.*, 2007, **11**, 1407–1416.
12. Dursun, U. Rotation hypersurfaces in Lorentz–Minkowski space with constant mean curvature. *Taiwanese J. Math.* (to appear).
13. Ki, U.-H., Kim, D.-S., Kim, Y. H., and Roh, Y.-M. Surfaces of revolution with pointwise 1-type Gauss map in Minkowski 3-space. *Taiwanese J. Math.* (to appear).
14. Kim, D.-S., Kim, Y. H., and Yoon, D. W. Extended B-scrolls and their Gauss map. *Indian J. Pure Appl. Math.*, 2002, **33**, 1031–1040.
15. Kim, Y. H. and Yoon, D. W. Ruled surfaces with finite type Gauss map in Minkowski spaces. *Soochow J. Math.*, 2000, **26**, 85–96.
16. Kim, Y. H. and Yoon, D. W. Ruled surfaces with pointwise 1-type Gauss map. *J. Geom. Phys.*, 2000, **34**, 191–205.

17. Kim, Y. H. and Yoon, D. W. Classifications of rotation surfaces in pseudo-Euclidean space. *J. Korean Math. Soc.*, 2004, **41**, 379–396.
18. Kim, Y. H. and Yoon, D. W. On the Gauss map of ruled surfaces in Minkowski space. *Rocky Mountain J. Math.*, 2005, **35**, 1555–1581.
19. Yoon, D. W. Rotation surfaces with finite type Gauss map in E^4. *Indian J. Pure. Appl. Math.*, 2001, **32**, 1803–1808.

Hüperpinnad punktiti 1-tüüpi Gaussi kujutusega Lorentzi-Minkowski ruumis

Uğur Dursun

On tõestatud, et orienteeritud hüperpind M_q indeksiga q Lorentzi-Minkowski ruumis $n+1$ on punktiti esimest liiki 1-tüüpi Gaussi kujutusega siis ja ainult siis, kui see M_q on konstantse keskmise kõverusega. Siit on järeldatud, et iga orienteeritud isoparameetriline hüperpind ruumis $n+1$ on 1-tüüpi Gaussi kujutusega. On klassifitseeritud kõik ratsionaalsed 1-tüüpi Gaussi kujutusega hüperpöördpinnad ruumis $n+1$ ja esitatud sellekohaseid näiteid.