A note on families of generalized Nörlund matrices as bounded operators on $l_p$

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Abstract. We deal with generalized Nörlund matrices $A = (N(p_n,q_n))$ defined by means of two non-negative sequences $(p_n)$ and $(q_n)$ with $p_0, q_0 > 0$. We are interested in simple conditions such that the associated non-negative triangular matrix $A = (a_{nk})$ is a bounded linear operator on $l_p$ ($1 < p < \infty$). Using results of D. Borwein (Canad. Math. Bull., 1998, 41, 10–14), we provide sufficient conditions and bounds for the norm $\|A\|_p$. Our main question is whether certain families of generalized Nörlund matrices $A_\alpha = (N(p_\alpha_n,q_n))$ studied by different authors (see, e.g., Anal. Math., 2003, 29, 227–242; Math. Z., 1993, 214, 273–286) are bounded linear operators on $l_p$. These matrices need not satisfy the sufficient conditions given by Borwein in the paper mentioned above. Explicit bounds for the norms $\|A_\alpha\|_p$ are given.

Key words: operator theory, Banach space $l_p$, bounded linear operators, generalized Nörlund matrices, Nörlund, Riesz and Euler–Knopp matrices.

1. INTRODUCTION AND PRELIMINARIES

1.1. Suppose throughout the paper that

$$1 < p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1.$$ 

Suppose also that $A = (a_{nk})$ is a triangular matrix of non-negative real numbers, that is, $a_{nk} \geq 0$ for $n,k \geq 0$, and $a_{nk} = 0$ for $n > k$, $n,k \in \mathbb{N}_0$. Let $l_p$ be the Banach space of all complex sequences $x = (x_n)$ ($n \in \mathbb{N}_0$) with the norm

$$\|x\|_p = \left( \sum_{n=0}^{\infty} |x_n|^p \right)^{1/p} < \infty,$$

and let $B(l_p)$ be the Banach algebra of all bounded linear operators on $l_p$. Thus $A \in B(l_p)$ if and only if $Ax \in l_p$ whenever $x \in l_p$, where $Ax = (y_n)$ with

$$(Ax)_n = y_n = \sum_{k=0}^{n} a_{nk}x_k.$$
Let
\[ \|A\|_p = \sup_{\|x\|_p \leq 1} \|Ax\|_p, \]
so that \( A \in B(l_p) \) if and only if \( \|A\|_p < \infty \), in which case \( \|A\|_p \) is the norm of \( A \).

It is well known that \( A \) is a bounded operator on the Banach space \( m \) of bounded sequences if and only if
\[ \sup_{n \in \mathbb{N}_0} \sum_{k=0}^n a_{nk} < \infty. \]

This condition, together with
\[ \lim_{n} a_{nk} = 0 \text{ for any } k \in \mathbb{N}_0, \]
is necessary and sufficient for \( A \) to be a bounded operator on the Banach space \( c_0 \). But even on these two conditions \( A \) need not be a bounded operator on \( l_p \). As an example the Nörlund method \( A = (N, e^{n\varphi}) \) with \( 0 < \varphi < 1 \) can be given (see [5]). Also, the Riesz weighted mean matrix \( A = (\bar{N}, \frac{1}{n+1}) \) is not a bounded operator on \( l_p \) because the necessary condition
\[ \sum_{n=0}^\infty (a_{nk})^p < \infty \quad (k \in \mathbb{N}_0) \]
for \( A \) to be bounded on \( l_p \) is not satisfied for it.

1.2. The problem of characterizing matrices in \( B(l_p) \) by means of conditions that are not complicated and difficult to apply has been discussed in a number of papers. This problem was discussed, for example, by D. Borwein and other mathematicians in papers [3,7,8] in general and, in particular, for Nörlund, Riesz weighted mean and Hausdorff matrices in \([1–6,10,12]\). In these papers different types of conditions (mostly sufficient) for \( A \) to be in \( B(l_p) \) were proved and illustrated with examples, also estimates for the norm \( \|A\|_p \) were found. It should be mentioned that already in 1943 G. H. Hardy proved (see [11]) an inequality which says that the Cesàro matrices \( A = (C, \alpha) \) and the Euler–Knopp matrices \( A = (E, \alpha) \) (\( \alpha > 0 \)) are in \( B(l_p) \) and that \( \|A\|_p = \frac{\Gamma(1+\alpha)^2}{\Gamma(\alpha+1)^2} \) and \( \|A\|_p = (\alpha + 1)^{1/p} \), respectively.

1.3. We consider in our paper generalized Nörlund matrices.

Suppose throughout the paper that \((p_n)\) and \((q_n)\) are two non-negative sequences such that \( p_0, q_0 > 0 \) and
\[ r_n = \sum_{k=0}^n p_{n-k} q_k \neq 0 \text{ for any } n \in \mathbb{N}_0. \]

Let us consider the generalized Nörlund matrix \( A = (N, p_n, q_n) \), i.e., the matrix \( A = (a_{nk}) \) with
\[ a_{nk} = \begin{cases} \frac{p_{n-k} q_k}{r_n} & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases} \]

In particular, if \( q_n = 1 \) for any \( n \in \mathbb{N}_0 \), then we get the Nörlund matrix \((N, p_n, 1) = (N, p_n)\). If \( p_n = 1 \) for any \( n \in \mathbb{N}_0 \), then we get the Riesz matrix \((N, 1, q_n) = (\bar{N}, q_n)\). In particular, if \( p_n = q_n^\alpha \) (\( \alpha > 0 \)) and \( q_n = \frac{1}{n^\alpha} \), we have the Euler–Knopp matrices \((N, p_n, q_n) = (E, \alpha)\).

The most convenient conditions to show that the matrix \( A = (N, p_n, q_n) \) is in \( B(l_p) \) come from the following theorem of D. Borwein (see [3], Theorem 2) proved for \( A = (a_{nk}) \).

**Theorem A.** Suppose that \( A = (a_{nk}) \) satisfies the conditions
\[ M_1 = \sup_{n \in \mathbb{N}_0} \sum_{k=0}^n a_{nk} < \infty \quad (1.1) \]
and
\[
a_{nk} \leq M_2 a_{nj} \quad \text{for} \quad 0 \leq k \leq j \leq n, \tag{1.2}
\]
where \(M_2\) is a positive number independent of \(k, j, n\).

Then \(A \in B(l_p)\) and
\[
\max \left\{ a_{00}, \frac{\lambda q}{M_2} \right\} \leq \|A\|_p \leq qM_1M_2^{q-1}, \tag{1.3}
\]
where \(\lambda = \liminf \frac{a_{nk}}{n}\).

Notice that \((N, p_n, q_n)\) satisfies (1.1) with \(M_1 = 1\). Thus Theorem A gives the following immediate corollary.

**Corollary 1.** If \((p_n)\) is non-increasing and \((q_n)\) is non-decreasing, then \(A = (N, p_n, q_n) \in B(l_p)\) and (1.3) holds with \(a_{00} = M_1 = M_2 = 1\).

**Example 1.** If \(A = (N, \frac{1}{n+1}, \log(n + 2))\) or \(A = (N, \frac{1}{n+1}, \log(n + 2))\), then \(A \in B(l_p)\) and \(\max \{1, \lambda, q\} \leq \|A\|_p \leq q\) by Corollary 1.

1.4. The main idea of our paper is to show that on the basis of a given matrix \(A = (N, p_n, q_n) \in B(l_p)\) the families of matrices \(A_\alpha\) being in \(B(l_p)\) can be constructed, where \(\alpha\) is a continuous or discrete parameter. Proving Theorems 1 and 2, we will find out some families of matrices \(A_\alpha = (N, p_n^\alpha, q_n)\) (see, e.g., [19] and [13]) which are in \(B(l_p)\) if \((N, p_n, q_n)\) is in \(B(l_p)\). It should be mentioned that if \(A = (N, p_n, q_n)\) satisfies the conditions of Corollary 1, then the matrices \(A_\alpha \in B(l_p)\) in Theorems 1 and 2 need not satisfy these conditions any more. In other words, \((p_n^\alpha)\) need not be non-increasing any more (if \((p_n)\) is), but nevertheless \(A_\alpha\) are bounded operators on \(l_p\).

1.5. We need also the preliminaries below.

The following theorem was published by Borwein in [3] as Theorem 1.

**Theorem B.** Suppose that \(A = (a_{nk})\) satisfies conditions (1.1),
\[
M_3 = \sup_{0 \leq k \leq n/2, n \in \mathbb{N}_0} (n + 1)a_{nk} < \infty, \tag{1.4}
\]
and
\[
M_4 = \sup_{k \in \mathbb{N}_0} \frac{2k}{n} a_{nk} < \infty. \tag{1.5}
\]
Then \(A \in B(l_p)\) and
\[
\|A\|_p \leq \mu_1^{1/p} \mu_2^{1/p}, \tag{1.6}
\]
where
\[
\mu_1 \leq 2^{1/p} M_1 + qM_3 \tag{1.7}
\]
and
\[
\mu_2 \leq M_4 + qM_3. \tag{1.8}
\]

We will use also the following simple proposition.

**Proposition A.** Let \(A_1\) and \(A_2\) be two matrices and \(A = A_2 A_1\) their product. If \(A_1 \in B(l_p)\) and \(A_2 \in B(l_p)\), then also \(A \in B(l_p)\) and
\[
\|A\|_p \leq \|A_2\|_p \|A_1\|_p. \tag{1.9}
\]
2. SOME REMARKS ON GENERALIZED NÖRLUND MATRICES \((N, p_n, q_n)\) AS BOUNDED OPERATORS ON \(l_p\)

2.1. First we notice that Corollary 1 can be slightly generalized.
If \((p_n)\) satisfies the condition
\[ C_1 a_n \leq p_n \leq C_2 a_n \quad (n \in \mathbb{N}_0), \tag{2.1} \]
where \((a_n)\) is some non-negative sequence and \(C_1\) and \(C_2\) are positive numbers not depending on \(n\), we write \(p_n \approx a_n\). If, in addition, \((a_n)\) is non-decreasing, then \((p_n)\) is said to be almost non-decreasing. If \(p_n \approx a_n\) and \((a_n)\) is non-increasing, then \((p_n)\) is said to be almost non-increasing.

Thus, if
\[ D_1 b_n \leq q_n \leq D_2 b_n \quad (n \in \mathbb{N}_0), \tag{2.2} \]
where \((b_n)\) is some non-decreasing sequence and \(D_1\) and \(D_2\) are positive constants, then \((q_n)\) is almost non-decreasing.

Now the following corollary from Theorem A improves Corollary 1.

**Corollary 2.** Suppose that \((p_n)\) is almost non-increasing and \((q_n)\) is almost non-decreasing, i.e., that (2.1) and (2.2) hold with some non-increasing \((a_n)\) and non-decreasing \((b_n)\), respectively. Then \(A = (N, p_n, q_n) \in B(l_p)\) and the estimate in (1.3) for the norm \(\|A\|_p\) is valid with \(M_1 = a_{00} = 1\) and \(M_2 = \frac{C_2 D_2}{C_1 D_1}\).

**Proof.** We have the inequalities
\[ p_n \geq C_1 a_n \geq C_1 a_j \geq \frac{C_1}{C_2} p_j \]
and
\[ q_n \leq D_2 b_n \leq D_2 b_j \leq \frac{D_2}{D_1} q_j \]
for any \(n \leq j\). Thus condition (1.2) is satisfied and our statement is true by Theorem A.

**Example 2.** If \(p_n = \frac{a^n}{n^r}\) and \(q_n = \log(n + 2)\), where \(\alpha > 0\), then \((N, p_n, q_n) \in B(l_p)\) by Corollary 2 because \((p_n)\) is almost non-increasing.

2.2. Applying Theorem B to \((N, p_n, q_n)\), we get the following result.

**Corollary 3.** Suppose that
\[ K_1 = \sup_{0 \leq k \leq n, n \in \mathbb{N}_0} \frac{q_k p_n}{r_n} < \infty \tag{2.3} \]
and
\[ K_2 = \sup_{n \in \mathbb{N}_0} \frac{(n + 1)p_n}{p_n} < \infty, \tag{2.4} \]
where \(P_n = \sum_{k=0}^{n} p_k\).

Then \(A = (N, p_n, q_n) \in B(l_p)\) and the norm \(\|A\|_p\) satisfies (1.6), where
\[ \mu_1 \leq 2^{1/p} + 2qK_1K_2 \tag{2.5} \]
and
\[ \mu_2 \leq K_1 + 2qK_1K_2. \tag{2.6} \]

**Proof.** Let us show that conditions (1.1), (1.4), and (1.5) are satisfied. We know that (1.1) is satisfied with \(M_1 = 1\). Further, with the help of (2.3) we get:
\[ \sum_{n-k}^{2k} a_{nk} \leq K_1 \sum_{n-k}^{2k} p_{n-k} p_n \leq \frac{K_1}{P_k} \sum_{n-k}^{2k} p_{n-k} = K_1. \]
Thus, (1.5) is satisfied with \( M_4 \leq K_1 \). Finally, using (2.3) and (2.4), we get for all \( 0 \leq k \leq n/2 \):

\[
(n + 1)a_{nk} = \frac{(n + 1)p_{n-k}q_n}{r_n} \leq K_1 \frac{(n + 1)p_{n-k}}{P_n} = K_1 \frac{(n + 1 - k)p_{n-k}}{P_n} \frac{P_{n-k}}{P_n} \frac{n + 1}{n - k + 1}
\]

\[
\leq K_1 K_2 \frac{2(n + 1)}{n + 2} \leq 2K_1 K_2.
\]

Thus also (1.4) is satisfied with \( M_3 \leq 2K_1 K_2 \). So we have by Theorem B that inequality (1.6) holds together with (2.5) and (2.6), which come from (1.7) and (1.8), respectively.

We add some remarks to Corollary 3.

**Remark 1.** In particular, if \( q_n = 1 \) for all \( n \in \mathbb{N}_0 \), then (2.3) is satisfied and \( K_1 = 1 \). For this partial case Corollary 3 was proved in [3] as Example 1.

**Example 3.** If \( p_n = 1 \ (n \in \mathbb{N}_0) \) and

\[
q_n = \begin{cases} 
1 & \text{if } n \text{ is even,} \\
0 & \text{if } n \text{ is odd,}
\end{cases}
\]

then \( A = (N, p_n, q_n) \in B(l_p) \) by Corollary 3 because conditions (2.3) and (2.4) are satisfied.

**Example 4.** Suppose that \( p_n \approx n^{\alpha-1}L_1(n) \) and \( q_n \approx n^\delta L_2(n) \), where \( \alpha > 0, \delta \geq 0, L_1(.) \) and \( L_2(.) \) are slowly varying functions and \( L_2(.) \) is non-decreasing. Let us show that \( A = (N, p_n, q_n) \in B(l_p) \). We have that \( (q_n) \) is almost non-decreasing,

\[
r_n \approx n^{\alpha+\delta}L_1(n)L_2(n)
\]

and

\[
P_n = \sum_{k=0}^{n} p_k \approx n^\alpha L_1(n)
\]

(see [13,15]. Thus (2.3) and (2.4) are satisfied and \( A \in B(l_p) \) by Corollary 3.

**Example 5.** If \( q_n = 1 \) and

\[
p_n = \begin{cases} 
1 & \text{if } n = m^2, \ m \in \mathbb{N}, \\
0 & \text{otherwise,}
\end{cases}
\]

then neither the conditions of Corollary 2 nor the conditions of Corollary 3 are satisfied but nevertheless \( (N, p_n, q_n) \in B(l_p) \) (see [2]).

2.3. The following corollary comes from Proposition A.

**Corollary 4.** Let \( A_1 = (N, p_n^1, q_n^1) \in B(l_p) \) and \( A_2 = (N, p_n^2, r_n^1) \in B(l_p) \).

(i) Then also \( A = (N, (p_2 * p_1)_n, q_n^1) \in B(l_p) \) and

\[
\|A\|_p \leq \|A_2\|_p \|A_1\|_p.
\]

(ii) In particular, if the sequences \( p_1 = (p_n^1) \) and \( p_2 = (p_n^2) \) are non-increasing and \( (q_n^1) \) is non-decreasing, then

\[
\|A\|_p \leq q^2.
\]
Let us suppose that

\[ A = (N, (p_2 * p_1)_n, q_n^1) = (N, p_n^2, r_n^1)(N, p_n^1, q_n^1), \]

then statement (i) is true by Proposition A and statement (ii) follows from (i) because by Corollary 1 we have for this particular case the inequalities \( ||A_1||_p \leq q \) and \( ||A_2||_p \leq q \).

3. SOME FAMILIES OF MATRICES BEING BOUNDED OPERATORS ON \( l_p \)

We consider here some families of matrices

\[ A_\alpha = (N, p_n^\alpha, q_n), \]

where \( \alpha \) is a continuous or discrete parameter. These families of matrices have been studied in different papers (see, e.g., \([9,13,14,16–20]\) on different levels of generality from the point of view of summability of sequences \( x = (x_n) \).

Applying Corollaries 2–4, we find the sufficient conditions for \( A_\alpha \in B(l_p) \) but do not focus on proving estimates for the norms \( ||A_\alpha||_p \).

**Theorem 1.** Let \( A_\alpha = (N, p_n^\alpha, q_n) \) be generalized Nörlund matrices, where \( \alpha \) is a continuous parameter with values \( \alpha > 0 \) and

\[ p_n^\alpha = \sum_{k=0}^{n} c_n^{\alpha} p_k, \]

where \( (c_n^{\alpha}) \) is either

(i) \( c_n^{\alpha} = A_n^{\alpha-1} = \left( \frac{n+1}{n} \right), \quad n \in \mathbb{N}_0, \)

or

(ii) \( c_n^{\alpha} = \frac{\alpha}{n}, \quad n \in \mathbb{N}_0. \)

If \( A = (N, p_n, q_n) \in B(l_p) \) and \( (r_n) \) is almost non-decreasing, then also \( (N, p_n^\alpha, q_n) \in B(l_p) \) for any \( \alpha > 0 \). In particular, if \( (r_n) \) is non-decreasing, then in case (i) the inequality \( ||A_\alpha||_p \leq q_n^{\alpha+1} ||A||_p \) holds, where \( [\alpha] \) is the integer part of \( \alpha \). More precisely, in this case \( ||A_\alpha||_p \leq q_0^{\alpha} ||A||_p \) if \( \alpha \in \mathbb{N} \).

We prove the theorem first for the special case if \( p_0 = 1 \) and \( p_n = 0 \) for any \( n \in \mathbb{N} \).

**Lemma.** Let us suppose that \( A_\alpha = (N, c_n^{\alpha}, q_n) \), where \( \alpha \) is a continuous parameter with values \( \alpha > 0 \), \( (q_n) \) is almost non-decreasing, and \( c_n^{\alpha} \) is defined as in Theorem 1 in both cases (i) and (ii). Then \( A_\alpha \in B(l_p) \) for any \( \alpha > 0 \).

In particular, if \( (q_n) \) is non-decreasing, then

\[ ||A_\alpha||_p \leq q^{[\alpha]+1} \quad (\alpha > 0) \]

(3.1)

in case (i). More precisely,

\[ ||A_\alpha||_p \leq q^\alpha \quad (\alpha \in \mathbb{N}). \]

(3.2)

**Proof.** For case (ii) notice that the sequence \( (c_n^{\alpha}) \) is almost non-increasing and thus \( A_\alpha \in B(l_p) \) by Corollary 2.

In case (i) we choose some \( \alpha > 0 \) and show that \( A_\alpha \in B(l_p) \) and that (3.1) and (3.2) hold in our particular case. If \( \alpha \leq 1 \), then \( c_n^{\alpha} = A_n^{\alpha-1} \) is non-increasing and our statement is true by Corollary 2.
If $\alpha > 1$, then $R_{n}^{\alpha} = \sum_{k=0}^{n} A_{n-k}^{\alpha-1} q_k$ is increasing. We use the equality

$$(N, A_{n}^{\alpha-1}, q_n) = (N, A_{n}^{\alpha-\delta-1}, r_{n}^{\delta}) (N, A_{n}^{\delta-1}, q_n)$$  \hspace{1cm} (\alpha > 0, \delta \geq 0)$$

(see, e.g., [19]). Taking $\delta = 1$, we can represent $A_{[\alpha]}$ in the form of the product

$$A_{[\alpha]} = (N, A_{n}^{0, r_{n}^{[\alpha]-1}}) \ldots (N, A_{n}^{2, r_{n}^{1}}) (N, A_{n}^{0, r_{n}^{1}}) (N, A_{n}^{1, r_{n}^{1}}, q_n).$$  \hspace{1cm} (3.3)$$

The right side of equality (3.3) is a product of $[\alpha]$ matrices. As $A_{n}^{0, r_{n}^{[\alpha]-1}}$, each of these matrices is in $B(l_{p})$ by Corollary 2 and therefore $A_{[\alpha]} \in B(l_{p})$ by Proposition A. In particular, if $(q_n)$ is non-decreasing, then each of the factors in the right side of equality (3.3) has a norm not greater than $q$ by Corollary 1. As a result, we get the inequality

$$\|A_{[\alpha]}\|_{p} \leq q^{[\alpha]}$$

in this particular case by Proposition A again. Thus, for $\alpha = [\alpha]$ our statement is proved. For $\alpha > 1$ in general we have the equality

$$A_{\alpha} = (N, A_{n}^{[\alpha]-[\alpha]-1}, r_{n}^{[\alpha]-1}) A_{[\alpha]}.$$  \hspace{1cm}$$

As both factors in the right side of the last equality are in $B(l_{p})$ and the norm of the first of them is not greater than $q$, $A_{\alpha}$ is in $B(l_{p})$, and also inequality (3.1) holds in the particular case by Proposition A.

**Proof of Theorem 1.** We have the equality

$$(N, p_{n}^{\alpha}, q_n) = (N, p_{n}^{\alpha}, r_{n}) (N, p_{n}, q_n)$$

for any $\alpha > 0$, where the right side is the product of matrices. As $(r_{n})$ is almost non-decreasing, $(N, l_{n}^{\alpha}, r_{n}) \in B(l_{p})$, and also (3.1) and (3.2) hold in the particular case by Lemma. Thus our statement is true by Proposition A.

**Example 6.** If $A = (N, p_{n}, q_n)$ is defined as in Examples 1, 2, 3, or 5, then $(N, p_{n}^{\alpha}, q_n) \in B(l_{p})$ for any $\alpha > 0$ by Theorem 1, because $(N, p_{n}, q_n) \in B(l_{p})$ and $(r_{n})$ is non-decreasing in these cases.

**Remark 2.** The best-known special cases of the matrices $(N, p_{n}^{\alpha}, q_n)$ given in Theorem 1 in case (i) are the Cesàro matrices $(C, \alpha)$, where $p_{n} = \delta_{n}$ and $q_{n} = 1$, and the generalized Cesàro matrices $(C, \alpha, \gamma)$, where $p_{n} = \delta_{n}$ and $q_{n} = (\alpha)$. An example of case (ii) is given by Euler–Knopp matrices $(E, \alpha)$ with $p_{n} = \delta_{n}$ and $q_{n} = 1/n!$.

**Theorem 2.** Consider the matrices $A_{\alpha} = (N, p_{n}^{\alpha}, q_n) = (N, p_{n}^{\alpha}, q_n)$ with the discrete parameter $\alpha \in \mathbb{N}$ defined by the convolutions $p_{n+1} = (p_{n})$ and $(p_{n}) = p_{n}^{\alpha} = (p_{n}^{\alpha}) = p_{n}^{\alpha} * p_{n}^{\alpha-1}$ for any $\alpha = 2, 3, \ldots$.

If

(i) $p_{n} \approx n^{\delta_{1}} L_{1}(n)$ and $q_{n} \approx n^{\delta_{2}} L_{2}(n)$, where $\delta_{1} > -1$, $\delta_{2} \geq 0$, $L_{1}(\cdot)$ and $L_{2}(\cdot)$ are slowly varying functions and $L_{2}(\cdot)$ is non-decreasing,

or

(ii) $(p_{n})$ is almost non-increasing and $(q_{n})$ is almost non-decreasing, then $A_{\alpha} \in B(l_{p})$ for any $\alpha \in \mathbb{N}$.

In particular, if $(p_{n})$ is non-increasing and $(q_{n})$ is non-decreasing, then $\|A_{\alpha}\|_{p} \leq q^{\alpha}$.

**Proof.** In case (i) we have $p_{n}^{\alpha} \approx n^{\delta_{1}+\alpha-1} L_{1}(n)$, where $L_{1}(\cdot)$ is also a slowly varying function (see [13]). Thus $A_{\alpha} \in B(l_{p})$ as was shown in Example 4.

In case (ii) we use the equality

$$(N, p_{n}^{\alpha+1}, q_n) = (N, p_{n}, r_{n}^{\alpha+1}) (N, p_{n}^{\alpha}, q_n),$$

where $r_{n}^{\alpha} = \sum_{k=0}^{n} p_{n-k}^{\alpha} q_k$ is almost non-decreasing because $(q_n)$ is almost non-decreasing. As $(N, p_{n}, q_n) \in B(l_{p})$ and $(N, p_{n}, q_{n}^{\alpha}) \in B(l_{p})$ for any $\alpha \in \mathbb{N}$ by Corollary 2, the relation $A_{\alpha} \in B(l_{p})$ and also the estimate of the norm $\|A_{\alpha}\|_{p}$ follow from Corollary 4 by induction.
Remark 3. We note that the matrices \( A_\alpha = (N, p_n^\alpha, q_n) \), which satisfy the conditions of Theorems 1 or 2 and are therefore bounded operators on \( l_p \), need not satisfy the conditions neither of Corollary 2 (Theorem A) nor of Corollary 3 (Theorem B). For example, if \( (p_n) \) is an almost non-increasing sequence, then \( (p_n^\alpha) \) need not be almost non-increasing any more. Moreover, \( (p_n^\alpha) \) is non-decreasing for any \( \alpha \geq 1 \) in case (i) of Theorem 1.

We finish our paper with an application of Corollary 3.

Theorem 3. Suppose that \( A_\alpha = (N, p_n^\alpha, q_n) \) (\( \alpha > 0 \)) are the same matrices as in Theorem 1 in case (i). If \( (p_n) \) satisfies condition (2.4), \( (q_n) \) is almost non-decreasing and satisfies the condition

\[
(n+1)q_n = O(Q_n),
\]

then \( A_\alpha \in B(l_p) \) for any \( \alpha > 0 \).

Proof. We apply Corollary 3 to the methods \( (N, p_n^\alpha, q_n) \) (instead of the methods \( (N, p_n, q_n) \)). We know that \( p_n^{\alpha+1} = \sum_{k=0}^{n} p_k^\alpha = O(n^\alpha P_n) \) and \( P_n Q_n = O(n^{1-\alpha} \alpha^{\alpha}) \) (see [18]). Thus condition (2.3) is satisfied:

\[
\frac{q_n p_n^{\alpha+1}}{n^\alpha} = O \left( \frac{q_n p_n^{\alpha+1}}{n^\alpha} \right) = O \left( \frac{Q_n p_n^{\alpha+1}}{(n+1) n^\alpha} \right) = O(1) \quad (k \leq n),
\]

and \( A_\alpha \in B(l_p) \) for any \( \alpha > 0 \) by Corollary 3.

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REFERENCES

Mõnest üldistatud Nörlundi maatriksite perest, kus maatriksid on tõkestatud operaatorid ruumis $l_p$

Ulrich Stadtmüller ja Anne Tali

On vaadeldud üldistatud Nörlundi maatrikseid $A = (N, p_n, q_n)$, mis on määratud kahe mittenegatiivse jadaga $(p_n)$ ja $(q_n)$, kus $p_0, q_0 > 0$. Vaatluse all on võimalikult lihtsad tingimused, selleks et maatriks $A$ oleks tõkestatud lineaarne operaator ruumis $l_p$ ($1 < p < \infty$). Kasutades artiklis [3] saadud tulemusi, on leitud eelmainitud piisavaid tingimusi, aga ka hinnanguid normi $\|A\|_p$ jaoks. Põhiprobleemina on uuritud, kas teatavad üldistatud Nörlundi maatriksite $A_\alpha = (N, p^{\alpha}_n, q_n)$ pered, mida on käsitletud mitmetes töödes (näiteks [19] ja [13]), moodustavad ruumis $l_p$ tõkestatud lineaarsete operaatorite pered. Need maatriksid ei tarvitse rahuldada artiklis [3] saadud piisavaid tingimusi.