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MATHEMATICS

A note on families of generalized Nörlund matrices as bounded operators on l_p

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Abstract. We deal with generalized Nörlund matrices $A=(N,p_n,q_n)$ defined by means of two non-negative sequences (p_n) and (q_n) with $p_0,q_0>0$. We are interested in simple conditions such that the associated non-negative triangular matrix $A=(a_{nk})$ is a bounded linear operator on l_p $(1<p<\infty)$. Using results of D. Borwein (*Canad. Math. Bull.*, 1998, **41**, 10–14), we provide sufficient conditions and bounds for the norm $\|A\|_p$. Our main question is whether certain families of generalized Nörlund matrices $A_\alpha=(N,p_n^\alpha,q_n)$ studied by different authors (see, e.g., *Anal. Math.*, 2003, **29**, 227–242; *Math. Z.*, 1993, **214**, 273–286) are bounded linear operators on l_p. These matrices need not satisfy the sufficient conditions given by Borwein in the paper mentioned above. Explicit bounds for the norms $\|A_\alpha\|_p$ are given.

Key words: operator theory, Banach space l_p, bounded linear operators, generalized Nörlund matrices, Nörlund, Riesz and Euler–Knopp matrices.

1. INTRODUCTION AND PRELIMINARIES

1.1. Suppose throughout the paper that

$$1<p<\infty,\quad \frac{1}{p}+\frac{1}{q}=1.$$

Suppose also that $A=(a_{nk})$ is a triangular matrix of non-negative real numbers, that is, $a_{nk}\geq 0$ for $n,k\geq 0$, and $a_{nk}=0$ for $n>k$, $n,k\in\mathbb{N}_0$. Let l_p be the Banach space of all complex sequences $x=(x_n)$ $(n\in\mathbb{N}_0)$ with the norm

$$\|x\|_p=\left(\sum_{n=0}^{\infty}|x_n|^p\right)^{1/p}<\infty,$$

and let $B(l_p)$ be the Banach algebra of all bounded linear operators on l_p. Thus $A\in B(l_p)$ if and only if $Ax\in l_p$ whenever $x\in l_p$, where $Ax=(y_n)$ with

$$(Ax)_n=y_n=\sum_{k=0}^{n}a_{nk}x_k.$$

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Let

$$\|A\|_p = \sup_{\|x\|_p \le 1} \|Ax\|_p,$$

so that $A \in B(l_p)$ if and only if $\|A\|_p < \infty$, in which case $\|A\|_p$ is the norm of A.

It is well known that A is a bounded operator on the Banach space m of bounded sequences if and only if

$$\sup_{n \in I\!N_0} \sum_{k=0}^{n} a_{nk} < \infty.$$

This condition, together with

$$\lim_n a_{nk} = 0 \text{ for any } k \in I\!N_0,$$

is necessary and sufficient for A to be a bounded operator on the Banach space c_0. But even on these two conditions A need not be a bounded operator on l_p. As an example the Nörlund method $A = (N, e^{n^\varphi})$ with $0 < \varphi < 1$ can be given (see [5]). Also, the Riesz weighted mean matrix $A = (\overline{N}, \frac{1}{n+1})$ is not a bounded operator on l_p because the necessary condition

$$\sum_{n=0}^{\infty} (a_{nk})^p < \infty \quad (k \in I\!N_0)$$

for A to be bounded on l_p is not satisfied for it.

1.2. The problem of characterizing matrices in $B(l_p)$ by means of conditions that are not complicated and difficult to apply has been discussed in a number of papers. This problem was discussed, for example, by D. Borwein and other mathematicians in papers [3,7,8] in general and, in particular, for Nörlund, Riesz weighted mean and Hausdorff matrices in [1–6,10,12]. In these papers different types of conditions (mostly sufficient) for A to be in $B(l_p)$ were proved and illustrated with examples, also estimates for the norm $\|A\|_p$ were found. It should be mentioned that already in 1943 G. H. Hardy proved (see [11]) an inequality which says that the Cesàro matrices $A = (C, \alpha)$ and the Euler–Knopp matrices $A = (E, \alpha)$ $(\alpha > 0)$ are in $B(l_p)$ and that $\|A\|_p = \frac{\Gamma(1+\alpha)\Gamma(1/q)}{\Gamma(\alpha+1/q)}$ and $\|A\|_p = (\alpha+1)^{1/p}$, respectively.

1.3. We consider in our paper generalized Nörlund matrices.

Suppose throughout the paper that (p_n) and (q_n) are two non-negative sequences such that $p_0, q_0 > 0$ and

$$r_n = \sum_{k=0}^{n} p_{n-k} q_k \ne 0 \text{ for any } n \in I\!N_0.$$

Let us consider the qeneralized Nörlund matrix $A = (N, p_n, q_n)$, i.e., the matrix $A = (a_{nk})$ with

$$a_{nk} = \begin{cases} \dfrac{p_{n-k}\, q_k}{r_n} & \text{if } 0 \le k \le n, \\ 0 & \text{if } k > n\,. \end{cases}$$

In particular, if $q_n = 1$ for any $n \in I\!N_0$, then we get the Nörlund matrix $(N, p_n, 1) = (N, p_n)$. If $p_n = 1$ for any $n \in I\!N_0$, then we get the Riesz matrix $(N, 1, q_n) = (\overline{N}, q_n)$. In particular, if $p_n = \frac{\alpha^n}{n!}$ $(\alpha > 0)$ and $q_n = \frac{1}{n!}$, we have the Euler–Knopp matrices $(N, p_n, q_n) = (E, \alpha)$.

The most convenient conditions to show that the matrix $A = (N, p_n, q_n)$ is in $B(l_p)$ come from the following theorem of D. Borwein (see [3], Theorem 2) proved for $A = (a_{nk})$.

Theorem A. *Suppose that $A = (a_{nk})$ satisfies the conditions*

$$M_1 = \sup_{n \in I\!N_0} \sum_{k=0}^{n} a_{nk} < \infty \tag{1.1}$$

and

$$a_{nk} \leq M_2 a_{nj} \quad for \quad 0 \leq k \leq j \leq n, \tag{1.2}$$

where M_2 is a positive number independent of k, j, n.

Then $A \in B(l_p)$ and

$$\max\left\{a_{00}, \frac{\lambda q}{M_2}\right\} \leq \|A\|_p \leq q M_1 M_2^{q-1}, \tag{1.3}$$

where $\lambda = \liminf n a_{n0}$.

Notice that (N, p_n, q_n) satisfies (1.1) with $M_1 = 1$. Thus Theorem A gives the following immediate corollary.

Corollary 1. *If (p_n) is non-increasing and (q_n) is non-decreasing, then $A = (N, p_n, q_n) \in B(l_p)$ and* (1.3) *holds with $a_{00} = M_1 = M_2 = 1$.*

Example 1. If $A = (N, \frac{1}{n+1}, \log(n+2))$ or $A = (N, \frac{1}{n!}, \log(n+2))$, then $A \in B(l_p)$ and $\max\{1, \lambda q\} \leq \|A\|_p \leq q$ by Corollary 1.

1.4. The main idea of our paper is to show that on the basis of a given matrix $A = (N, p_n, q_n) \in B(l_p)$ the families of matrices A_α being in $B(l_p)$ can be constructed, where α is a continuous or discrete parameter. Proving Theorems 1 and 2, we will find out some families of matrices $A_\alpha = (N, p_n^\alpha, q_n)$ (see, e.g., [19] and [13]) which are in $B(l_p)$ if (N, p_n, q_n) is in $B(l_p)$. It should be mentioned that if $A = (N, p_n, q_n)$ satisfies the conditions of Corollary 1, then the matrices $A_\alpha \in B(l_p)$ in Theorems 1 and 2 need not satisfy these conditions any more. In other words, (p_n^α) need not be non-increasing any more (if (p_n) is), but nevertheless A_α are bounded operators on l_p.

1.5. We need also the preliminaries below.

The following theorem was published by Borwein in [3] as Theorem 1.

Theorem B. *Suppose that $A = (a_{nk})$ satisfies conditions* (1.1),

$$M_3 = \sup_{0 \leq k \leq n/2, n \in \mathbb{N}_0} (n+1) a_{nk} < \infty, \tag{1.4}$$

and

$$M_4 = \sup_{k \in \mathbb{N}_0} \sum_{n=k}^{2k} a_{nk} < \infty. \tag{1.5}$$

Then $A \in B(l_p)$ and

$$\|A\|_p \leq \mu_1^{1/q} \mu_2^{1/p}, \tag{1.6}$$

where

$$\mu_1 \leq 2^{1/p} M_1 + q M_3 \tag{1.7}$$

and

$$\mu_2 \leq M_4 + q M_3. \tag{1.8}$$

We will use also the following simple proposition.

Proposition A. *Let A_1 and A_2 be two matrices and $A = A_2 A_1$ their product. If $A_1 \in B(l_p)$ and $A_2 \in B(l_p)$, then also $A \in B(l_p)$ and*

$$\|A\|_p \leq \|A_2\|_p \|A_1\|_p.$$

2. SOME REMARKS ON GENERALIZED NÖRLUND MATRICES (N, p_n, q_n) AS BOUNDED OPERATORS ON l_p

2.1. First we notice that Corollary 1 can be slightly generalized.

If (p_n) satisfies the condition

$$C_1 a_n \leq p_n \leq C_2 a_n \quad (n \in \mathbb{N}_0), \tag{2.1}$$

where (a_n) is some non-negative sequence and C_1 and C_2 are positive numbers not depending on n, we write $p_n \approx a_n$. If, in addition, (a_n) is non-decreasing, then (p_n) is said to be almost non-decreasing. If $p_n \approx a_n$ and (a_n) is non-increasing, then (p_n) is said to be almost non-increasing.

Thus, if

$$D_1 b_n \leq q_n \leq D_2 b_n \quad (n \in \mathbb{N}_0), \tag{2.2}$$

where (b_n) is some non-decreasing sequence and D_1 and D_2 are positive constants, then (q_n) is almost non-decreasing.

Now the following corollary from Theorem A improves Corollary 1.

Corollary 2. *Suppose that (p_n) is almost non-increasing and (q_n) is almost non-decreasing, i.e., that* (2.1) *and* (2.2) *hold with some non-increasing (a_n) and non-decreasing (b_n), respectively. Then $A = (N, p_n, q_n) \in B(l_p)$ and the estimate in* (1.3) *for the norm $\|A\|_p$ is valid with $M_1 = a_{00} = 1$ and $M_2 = \frac{C_2 D_2}{C_1 D_1}$.*

Proof. We have the inequalities

$$p_n \geq C_1 a_n \geq C_1 a_j \geq \frac{C_1}{C_2} p_j$$

and

$$q_n \leq D_2 b_n \leq D_2 b_j \leq \frac{D_2}{D_1} q_j$$

for any $n \leq j$. Thus condition (1.2) is satisfied and our statement is true by Theorem A.

Example 2. If $p_n = \frac{\alpha^n}{n!}$ and $q_n = \log(n+2)$, where $\alpha > 0$, then $(N, p_n, q_n) \in B(l_p)$ by Corollary 2 because (p_n) is almost non-increasing.

2.2. Applying Theorem B to (N, p_n, q_n), we get the following result.

Corollary 3. *Suppose that*

$$K_1 = \sup_{0 \leq k \leq n, n \in \mathbb{N}_0} \frac{q_k P_n}{r_n} < \infty \tag{2.3}$$

and

$$K_2 = \sup_{n \in \mathbb{N}_0} \frac{(n+1) p_n}{P_n} < \infty, \tag{2.4}$$

where $P_n = \sum_{k=0}^{n} p_k$.

Then $A = (N, p_n, q_n) \in B(l_p)$ and the norm $\|A\|_p$ satisfies (1.6), *where*

$$\mu_1 \leq 2^{1/p} + 2q K_1 K_2 \tag{2.5}$$

and

$$\mu_2 \leq K_1 + 2q K_1 K_2. \tag{2.6}$$

Proof. Let us show that conditions (1.1), (1.4), and (1.5) are satisfied. We know that (1.1) is satisfied with $M_1 = 1$. Further, with the help of (2.3) we get:

$$\sum_{n=k}^{2k} a_{nk} \leq K_1 \sum_{n=k}^{2k} \frac{p_{n-k}}{P_n} \leq \frac{K_1}{P_k} \sum_{n=k}^{2k} p_{n-k} = K_1.$$

Thus, (1.5) is satisfied with $M_4 \le K_1$. Finally, using (2.3) and (2.4), we get for all $0 \le k \le n/2$:

$$\begin{aligned}(n+1)a_{nk} &= \frac{(n+1)p_{n-k}q_k}{r_n} \le K_1 \frac{(n+1)p_{n-k}}{P_n} = K_1 \frac{(n+1-k)p_{n-k}}{P_{n-k}} \frac{P_{n-k}}{P_n} \frac{n+1}{n-k+1} \\ &\le K_1 K_2 \frac{2(n+1)}{n+2} \le 2K_1 K_2.\end{aligned}$$

Thus also (1.4) is satisfied with $M_3 \le 2K_1K_2$. So we have by Theorem B that inequality (1.6) holds together with (2.5) and (2.6), which come from (1.7) and (1.8), respectively.

We add some remarks to Corollary 3.

Remark 1. In particular, if $q_n = 1$ for all $n \in \mathbb{N}_0$, then (2.3) is satisfied and $K_1 = 1$. For this partial case Corollary 3 was proved in [3] as Example 1.

Example 3. If $p_n = 1 \quad (n \in \mathbb{N}_0)$ and

$$q_n = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

then $A = (N, p_n, q_n) \in B(l_p)$ by Corollary 3 because conditions (2.3) and (2.4) are satisfied.

Example 4. Suppose that $p_n \approx n^{\alpha-1}L_1(n)$ and $q_n \approx n^{\delta}L_2(n)$, where $\alpha > 0$, $\delta \ge 0$, $L_1(.)$ and $L_2(.)$ are slowly varying functions and $L_2(.)$ is non-decreasing. Let us show that $A = (N, p_n, q_n) \in B(l_p)$. We have that (q_n) is almost non-decreasing,

$$r_n \approx n^{\alpha+\delta} L_1(n) L_2(n)$$

and

$$P_n = \sum_{k=0}^{n} p_k \approx n^{\alpha} L_1(n)$$

(see [13,15]. Thus (2.3) and (2.4) are satisfied and $A \in B(l_p)$ by Corollary 3.

Example 5. If $q_n = 1$ and

$$p_n = \begin{cases} 1 & \text{if } n = m^2,\ m \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

then neither the conditions of Corollary 2 nor the conditions of Corollary 3 are satisfied but nevertheless $(N, p_n, q_n) \in B(l_p)$ (see [2]).

2.3. The following corollary comes from Proposition A.

Corollary 4. *Let $A_1 = (N, p_n^1, q_n^1) \in B(l_p)$ and $A_2 = (N, p_n^2, r_n^1) \in B(l_p)$.*

(i) *Then also $A = (N, (p_2 * p_1)_n, q_n^1) \in B(l_p)$ and*

$$\|A\|_p \le \|A_2\|_p \, \|A_1\|_p.$$

(ii) *In particular, if the sequences $p_1 = (p_n^1)$ and $p_2 = (p_n^2)$ are non-increasing and (q_n^1) is non-decreasing, then*

$$\|A\|_p \le q^2.$$

Proof. As A is the product of matrices

$$A = (N, (p_2 * p_1)_n, q_n^1) = (N, p_n^2, r_n^1)\,(N, p_n^1, q_n^1),$$

then statement (i) is true by Proposition A and statement (ii) follows from (i) because by Corollary 1 we have for this particular case the inequalities $\|A_1\|_p \le q$ and $\|A_2\|_p \le q$.

3. SOME FAMILIES OF MATRICES BEING BOUNDED OPERATORS ON l_p

We consider here some families of matrices

$$A_\alpha = (N, p_n^\alpha, q_n),$$

where α is a continuous or discrete parameter. These families of matrices have been studied in different papers (see, e.g., [9,13,14,16–20] on different levels of generality from the point of view of summability of sequences $x = (x_n)$.

Applying Corollaries 2–4, we find the sufficient conditions for $A_\alpha \in B(l_p)$ but do not focus on proving estimates for the norms $\|A_\alpha\|_p$.

Theorem 1. *Let $A_\alpha = (N, p_n^\alpha, q_n)$ be generalized Nörlund matrices, where α is a continuous parameter with values $\alpha > 0$ and*

$$p_n^\alpha = \sum_{k=0}^{n} c_{n-k}^\alpha\, p_k,$$

where (c_n^α) is either

(i) $c_n^\alpha = A_n^{\alpha-1} = \binom{n+\alpha-1}{n}, \quad n \in \mathbb{N}_0,$

or

(ii) $c_n^\alpha = \frac{\alpha^n}{n!}, \quad n \in \mathbb{N}_0.$

If $A = (N, p_n, q_n) \in B(l_p)$ and (r_n) is almost non-decreasing, then also $(N, p_n^\alpha, q_n) \in B(l_p)$ for any $\alpha > 0$. In particular, if (r_n) is non-decreasing, then in case (i) *the inequality $\|A_\alpha\|_p \le q^{[\alpha]+1}\|A\|_p$ holds, where $[\alpha]$ is the integer part of α. More precisely, in this case $\|A_\alpha\|_p \le q^\alpha \|A\|_p$ if $\alpha \in \mathbb{N}$.*

We prove the theorem first for the special case if $p_0 = 1$ and $p_n = 0$ for any $n \in \mathbb{N}$.

Lemma. *Let us suppose that $A_\alpha = (N, c_n^\alpha, q_n)$, where α is a continuous parameter with values $\alpha > 0$, (q_n) is almost non-decreasing, and c_n^α is defined as in Theorem* 1 *in both cases* (i) *and* (ii). *Then $A_\alpha \in B(l_p)$ for any $\alpha > 0$.*

In particular, if (q_n) is non-decreasing, then

$$\|A_\alpha\|_p \le q^{[\alpha]+1} \quad (\alpha > 0) \tag{3.1}$$

in case (i). *More precisely,*

$$\|A_\alpha\|_p \le q^\alpha \quad (\alpha \in \mathbb{N}). \tag{3.2}$$

Proof. For case (ii) notice that the sequence (c_n^α) is almost non-increasing and thus $A_\alpha \in B(l_p)$ by Corollary 2.

In case (i) we choose some $\alpha > 0$ and show that $A_\alpha \in B(l_p)$ and that (3.1) and (3.2) hold in our particular case. If $\alpha \le 1$, then $c_n^\alpha = A_n^{\alpha-1}$ is non-increasing and our statement is true by Corollary 2.

If $\alpha > 1$, then $r_n^\alpha = \sum_{k=0}^n A_{n-k}^{\alpha-1} q_k$ is increasing. We use the equality

$$(N, A_n^{\alpha-1}, q_n) = (N, A_n^{\alpha-\delta-1}, r_n^\delta)\,(N, A_n^{\delta-1}, q_n) \quad (\alpha > 0, \delta \geq 0)$$

(see, e.g., [19]). Taking $\delta = 1$, we can represent $A_{[\alpha]}$ in the form of the product

$$A_{[\alpha]} = (N, A_n^0, r_n^{[\alpha]-1})...(N, A_n^0, r_n^2)\,(N, A_n^0, r_n^1)\,(N, A_n^{1-1}, q_n). \tag{3.3}$$

The right side of equality (3.3) is a product of $[\alpha]$ matrices. As $A_n^0 = 1$, each of these matrices is in $B(l_p)$ by Corollary 2 and therefore $A_{[\alpha]} \in B(l_p)$ by Proposition A. In particular, if (q_n) is non-decreasing, then each of the factors in the right side of equality (3.3) has a norm not greater than q by Corollary 1. As a result, we get the inequality

$$\|A_{[\alpha]}\|_p \leq q^{[\alpha]}$$

in this particular case by Proposition A again. Thus, for $\alpha = [\alpha]$ our statement is proved. For $\alpha > 1$ in general we have the equality

$$A_\alpha = (N, A_n^{\alpha-[\alpha]-1}, r_n^{[\alpha]})A_{[\alpha]}.$$

As both factors in the right side of the last equality are in $B(l_p)$ and the norm of the first of them is not greater than q, A_α is in $B(l_p)$, and also inequality (3.1) holds in the particular case by Proposition A.

Proof of Theorem 1. We have the equality

$$(N, p_n^\alpha, q_n) = (N, c_n^\alpha, r_n)(N, p_n, q_n)$$

for any $\alpha > 0$, where the right side is the product of matrices. As (r_n) is almost non-decreasing, $(N, c_n^\alpha, r_n) \in B(l_p)$, and also (3.1) and (3.2) hold in the particular case by Lemma. Thus our statement is true by Proposition A.

Example 6. If $A = (N, p_n, q_n)$ is defined as in Examples 1, 2, 3, or 5, then $(N, p_n^\alpha, q_n) \in B(l_p)$ for any $\alpha > 0$ by Theorem 1, because $(N, p_n, q_n) \in B(l_p)$ and (r_n) is non-decreasing in these cases.

Remark 2. The best-known special cases of the matrices (N, p_n^α, q_n) given in Theorem 1 in case (i) are the Cesàro matrices (C, α), where $p_n = \delta_{0n}$ and $q_n = 1$, and the generalized Cesàro matrices (C, α, γ), where $p_n = \delta_{0n}$ and $q_n = \binom{n+\gamma}{n}$. An example of case (ii) is given by Euler–Knopp matrices (E, α) with $p_n = \delta_{0n}$ and $q_n = 1/n!$.

Theorem 2. *Consider the matrices $A_\alpha = (N, p_n^\alpha, q_n) = (N, p_n^{*\alpha}, q_n)$ with the discrete parameter $\alpha \in \mathbb{N}$ defined by the convolutions $p^{*1} = (p_n)$ and $(p_n^\alpha) = p^{*\alpha} = (p_n^{*\alpha}) = p^{*1} * p^{*(\alpha-1)}$ for any $\alpha = 2, 3, \ldots$.*

If

(i) *$p_n \approx n^{\delta_1} L_1(n)$ and $q_n \approx n^{\delta_2} L_2(n)$, where $\delta_1 > -1$, $\delta_2 \geq 0$, $L_1(.)$ and $L_2(.)$ are slowly varying functions and $L_2(.)$ is non-decreasing,*

or

(ii) *(p_n) is almost non-increasing and (q_n) is almost non-decreasing, then $A_\alpha \in B(l_p)$ for any $\alpha \in \mathbb{N}$. In particular, if (p_n) is non-increasing and (q_n) is non-decreasing, then $\|A_\alpha\|_p \leq q^\alpha$.*

Proof. In case (i) we have $p_n^{*\alpha} \approx n^{\alpha\delta_1+\alpha-1} L_1^\alpha(n)$, where $L_1^\alpha(.)$ is also a slowly varying function (see [13]). Thus $A_\alpha \in B(l_p)$ as was shown in Example 4.

In case (ii) we use the equality

$$(N, p_n^{*(\alpha+1)}, q_n) = (N, p_n, r_n^\alpha)(N, p_n^{*\alpha}, q_n),$$

where $r_n^\alpha = \sum_{k=0}^n p_{n-k}^{*\alpha} q_k$ is almost non-decreasing because (q_n) is almost non-decreasing. As $(N, p_n, q_n) \in B(l_p)$ and $(N, p_n, r_n^\alpha) \in B(l_p)$ for any $\alpha \in \mathbb{N}$ by Corollary 2, the relation $A_\alpha \in B(l_p)$ and also the estimate of the norm $\|A_\alpha\|_p$ follow from Corollary 4 by induction.

Remark 3. We note that the matrices $A_\alpha = (N, p_n^\alpha, q_n)$, which satisfy the conditions of Theorems 1 or 2 and are therefore bounded operators on l_p, need not satisfy the conditions neither of Corollary 2 (Theorem A) nor of Corollary 3 (Theorem B). For example, if (p_n) is an almost non-increasing sequence, then (p_n^α) need not be almost non-increasing any more. Moreover, (p_n^α) is non-decreasing for any $\alpha \geq 1$ in case (i) of Theorem 1.

We finish our paper with an application of Corollary 3.

Theorem 3. *Suppose that* $A_\alpha = (N, p_n^\alpha, q_n)$ $(\alpha > 0)$ *are the same matrices as in Theorem* 1 *in case* (i)*. If* (p_n) *satisfies condition* (2.4), (q_n) *is almost non-decreasing and satisfies the condition*

$$(n+1)q_n = O(Q_n),$$

then $A_\alpha \in B(l_p)$ *for any* $\alpha > 0$.

Proof. We apply Corollary 3 to the methods (N, p_n^α, q_n) (instead of the methods (N, p_n, q_n)). We know that $p_n^{\alpha+1} = \sum_{k=0}^n p_k^\alpha = O(n^\alpha P_n)$ and $P_n Q_n = O(n^{1-\alpha} r_n^\alpha)$ (see [18]). Thus condition (2.3) is satisfied:

$$\frac{q_k p_n^{\alpha+1}}{r_n^\alpha} = O\left(\frac{q_n p_n^{\alpha+1}}{r_n^\alpha}\right) = O\left(\frac{Q_n P_n n^\alpha}{(n+1) r_n^\alpha}\right) = O(1) \quad (k \leq n),$$

and $A_\alpha \in B(l_p)$ for any $\alpha > 0$ by Corollary 3.

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Mõnest üldistatud Nörlundi maatriksite perest, kus maatriksid on tõkestatud operaatorid ruumis l_p

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On vaadeldud üldistatud Nörlundi maatrikseid $A = (N, p_n, q_n)$, mis on määratud kahe mittenegatiivse jadaga (p_n) ja (q_n), kus $p_0, q_0 > 0$. Vaatluse all on võimalikult lihtsad tingimused, selleks et maatriks A oleks tõkestatud lineaarne operaator ruumis l_p $(1 < p < \infty)$. Kasutades artiklis [3] saadud tulemusi, on leitud eelmainitud piisavaid tingimusi, aga ka hinnanguid normi $\|A\|_p$ jaoks. Põhiprobleemina on uuritud, kas teatavad üldistatud Nörlundi maatriksite $A_\alpha = (N, p_n^\alpha, q_n)$ pered, mida on käsitletud mitmetes töödes (näiteks [19] ja [13]), moodustavad ruumis l_p tõkestatud lineaarsete operaatorite pered. Need maatriksid ei tarvitse rahuldada artiklis [3] saadud piisavaid tingimusi.